On integrable deformation of the Poincaré system

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An integrable deformation of the Poincaré system is considered. This system is derived by using the general $r$-matrix theory.

1. Introduction

One of the cornerstones of the quantum and classical inverse scattering method is the $r$-matrix algebras with a spectral parameter dependence. The rich structure of such algebras allows us to cover with this formalism a variety of known integrable models and to find new ones.

In this paper we consider representation of the Sklyanin $r$-matrix algebra on symplectic leaves of the Lie algebra $so(4) = so(3) \oplus so(3)$ using standard techniques of the reflection equation theory. As a matter of fact, the construction leads automatically to an integrable system with additional quartic integral of motion.

Let $s_i, t_i$, $i = 1, 2, 3$, be coordinates on the algebra $so(4) = so(3) \oplus so(3)$ with the following Lie-Poisson brackets

\[
\{s_i, s_j\} = \varepsilon_{ijk} s_k, \quad \{s_i, t_j\} = 0, \quad \{t_i, t_j\} = \varepsilon_{ijk} t_k.
\]  

(1.1)

The generic symplectic leaves

\[O_{ab} : \quad \{(s, t) : \quad C_1 = a, \quad C_2 = b\}\]

(1.2)

are specified by the fixed values of two Casimir elements

\[C_1 = s^2, \quad C_2 = t^2,\]

(1.3)

where $s$ and $t$ are vectors with components $s_i, t_i$.

2. The Lax matrices on $so(4)$

Let us begin with the known Lax matrix associated with the usual two-site $XXX$ Heisenberg magnet

\[
T(\lambda) = \begin{pmatrix}
\lambda - s_3 + i p_1 & s_1 + i s_2 & \lambda - t_3 + i p_2 & t_1 + i t_2 \\
\lambda + s_3 + i p_1 & t_1 - i t_2 & \lambda + t_3 + i p_2 \\
\end{pmatrix}
\]

(2.1)

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depending on two arbitrary parameters \( p_{1,2} \in \mathbb{R} \). This matrix \( T(\lambda) \) (2.1) defines a representation of the Sklyanin algebra

\[
\{ T(\lambda), T(\nu) \} = \left[ r(\lambda - \nu), T(\lambda)T(\nu) \right],
\]

(2.2)
on generic symplectic leaves \( O_{ab} \) (1.2) of the algebra \( so(4) \). Here we use the standard notation \( \frac{1}{T} = T(\lambda) \otimes Id, \frac{2}{T} = Id \otimes T(\nu) \) and \( r \)-matrix has the form

\[
r(\lambda - \nu) = \frac{i}{\lambda - \nu} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(2.3)

The main property of the Sklyanin algebra (2.2) is that for any matrix with numerical entries \( \mathcal{K} \) the coefficients of the trace of the matrix \( \mathcal{K} T(\lambda) \) give rise to the commutative subalgebra

\[
\{ \text{tr} \mathcal{K} T(\lambda), \text{tr} \mathcal{K} T(\nu) \} = 0.
\]

All the generators of this subalgebra are linear polynomials with coefficients composed of the entries \( T_{ij}(\lambda) \). The entries polynomials are interpreted as integrals of motion for the integrable system associated with the matrix \( T(\lambda) \). For instance, the representation (2.1) generates one linear and one quadratic integrals of motion in variables \( s_i, t_i \) and the corresponding integrable system on \( O_{ab} \) is equivalent to a special case of the Poincaré system [3].

According to [4], we can construct another commutative subalgebra generated by quadratic polynomials with coefficients composed of \( T_{ij}(\lambda) \), which are integrals of motion for another integrable system associated with the same matrix \( T(\lambda) \). Recall that if the matrices \( \mathcal{K}_\pm(\lambda) \) are solutions to the reflection equation

\[
\{ \mathcal{K}(\lambda), \mathcal{K}(\nu) \} = \left[ r(\lambda - \nu), \mathcal{K}(\lambda)\mathcal{K}(\nu) \right] + \mathcal{K}(\lambda) r(\lambda + \nu) \mathcal{K}(\nu) r(\lambda + \nu) \mathcal{K}(\lambda),
\]

(2.4)

then the coefficients of the trace of the Lax matrix

\[
L(\lambda) = \mathcal{K}_-(\lambda) T(\lambda) \mathcal{K}_+(\lambda) \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} T^t(-\lambda) \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]

(2.5)
give rise to the commutative subalgebra

\[
\{ \text{tr} L(\lambda), \text{tr} L(\nu) \} = 0.
\]

In (2.5) the superscript \( t \) stands for matrix transposition, the matrix \( T(\lambda) \) satisfies (2.2) and commutes with \( \mathcal{K}(\lambda) \).

So, using the known representation \( T(\lambda) \) (2.1) of the Sklyanin algebra (2.2) and the two general numerical solutions \( \mathcal{K}_\pm(\lambda) \) to the reflection equation

\[
\mathcal{K}_+ = \begin{pmatrix}
 a_1 \lambda + a_0 & (b_1 + ic_1)\lambda \\
 (b_1 - ic_1)\lambda & -a_1 \lambda + a_0
\end{pmatrix}, \quad \mathcal{K}_- = \begin{pmatrix}
 d_1 \lambda + d_0 & (b_0 + ic_0)\lambda \\
 (b_0 - ic_0)\lambda & -d_1 \lambda + d_0
\end{pmatrix},
\]

one gets a new Lax matrix \( L(\lambda) \) (2.5) on the generic symplectic leaves \( O_{ab} \) of \( so(4) \).

The determinant of this new Lax matrix \( L(\lambda) \) (2.5)

\[
\det L(\lambda) = \left( a_0^2 - \lambda^2(a_1^2 + b_1^2 + c_1^2) \right) \left( \lambda^4 + 2(p_2^2 - C_1)\lambda^2 + (p_1^2 + C_1)^2 \right) \times \left( d_0^2 - \lambda^2(d_1^2 + b_0^2 + c_0^2) \right) \left( \lambda^4 + 2(p_2^2 - C_2)\lambda^2 + (p_2^2 + C_2)^2 \right)
\]

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depends only on the Casimir operators (1.3) and numerical parameters \(a_i, b_i, c_i, d_i\) and \(p_i\). The trace of the Lax matrix \(L(\lambda)\) (2.5)

\[
\text{tr} \ L(\lambda) = -2\lambda^6(b_0b_1 + c_0c_1 + a_1d_1) - 2\lambda^4(H + a_0d_0 + (b_0b_1 + c_0c_1 + a_1d_1)(p_1^2 + p_2^2)) \\
- 2\lambda^2K - 2a_0d_0(p_1^2 + C_1)(p_2^2 + C_2).
\]

(2.6)

gives rise to the integrals of motion, \(H\) and \(K\), which are in involution

\[\{H, K\} = 0.\]

The integrals \(H\) and \(K\) are second and fourth order polynomials in variables \(s_i\) and \(t_i\), respectively.

The Hamilton function \(H\) in (2.6) is a real-valued function in the physical variables \(s_i\) and \(t_i\)

\[H = (s, As) + 4(s, Bt) + (t, At) + 2(a, s + t) + 2(b, p_1s + p_2t)\]

(2.7)

and depending on the parameters \(a_1, \ldots, d_1\) and \(p_i\). Here \(A\) is a symmetric matrix, that is,

\[A = \begin{pmatrix}
b_0b_1 - c_0c_1 - a_1d_1 & b_1c_0 + b_0c_1 & -a_1b_0 - b_1d_1 \\
b_1c_0 + b_0c_1 & c_0c_1 - b_0b_1 - a_1d_1 & -a_1c_0 - c_1d_1 \\
-a_1b_0 - b_1d_1 & -a_1c_0 - c_1d_1 & a_1d_1 - b_0b_1 - c_0c_1 \\
\end{pmatrix}.
\]

The matrix \(B\) reads

\[B = \begin{pmatrix}
b_0b_1 & c_0b_1 & -d_1b_1 \\
b_0c_1 & c_0c_1 & -d_1c_1 \\
-b_0a_1 & -c_0a_1 & d_1a_1 \\
\end{pmatrix}, \quad \text{det} \ B = 0,
\]

and the vectors \(a\) and \(b\) are

\[a = (a_0b_0 + b_1d_0, a_0c_0 + c_1d_0, -a_1d_0 - a_0d_1), \quad (2.8)
\]

\[b = (-a_1c_0 + c_1d_1, a_1b_0 - b_1d_1, -b_1c_0 + b_0c_1).\]

The matrix \(A\) does not commute with the matrix \(B\). Hence, using the elements of the group \(SO(3) \times SO(3)\) we can not bring simultaneously both matrices \(A\) and \(B\) to diagonal form.

Namely, after a suitable rotation of the vectors \(s\) and \(t\), the matrices \(A, B\) and the vectors \(a, b\) may be rewritten in the following form

\[A' = V^tAV = \begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & -\alpha_2
\end{pmatrix}, \quad B' = V^tBV = \begin{pmatrix}
0 & 0 & 0 \\
0 & \beta_1 & \beta_2 \\
0 & \beta_3 & \beta_4
\end{pmatrix},
\]

and

\[a' = V^ta = (0, \gamma_1, \gamma_2), \quad b' = V^tb = (\delta, 0, 0).
\]

Here \(V\) is an orthogonal matrix, i.e. \(V^tV = VV^t = I_d\) and the parameters \(\alpha_{1,2}, \beta_{1,4}, \gamma_{1,2}\) and \(\delta\) are functions of the eight initial parameters \(a_{0,1}, b_{0,1}, c_{0,1}\) and \(d_{0,1}\).

In closing this section, let us compare (2.7) with the known examples of integrable Hamiltonians on \(so(4)\) (see Table 3.2 in [1] and [6]). Recall that, there is a Poincaré system with an additional linear integral and two families (the so-called Manakov and Steklov cases) characterized with the property that there exists an additional quadratic integral. In the fourth known integrable case (the so-called Adler and van Moerbeke case) the additional integral is quartic. In contrast to the
Hamiltonian (2.7), for all these systems the matrices \( A \) and \( B \) commute and can be reduced to a diagonal form simultaneously.

Recently, some deformation of the Kowalevski gyrostat on the algebra \( so(4) \) has been proposed by Sokolov [5]. In this case, the additional integral is quartic and the corresponding matrices \( A \) and \( B \) do not commute as well. In order to compare (2.7) with this system we consider a matrix representation of \( so(4) \) in the next section.

### 3. Matrix representation of \( so(4) \)

For any parameter \( \varkappa \neq 0 \) the standard mapping

\[
x = \varkappa (s - t), \quad J = s + t
\]

defines another representation of the complex algebra \( so(4, \mathbb{C}) \) with the following Lie–Poisson brackets

\[
\{ J_i, J_j \} = \varepsilon_{ijk} J_k, \quad \{ J_i, x_j \} = \varepsilon_{ijk} x_k, \quad \{ x_i, x_j \} = \varkappa^2 \varepsilon_{ijk} J_k, \tag{3.1}
\]

The corresponding Casimir elements are

\[
C_{\varkappa} = \varkappa^2 J^2 + x^2, \quad \ell_{\varkappa} = (x, J).
\]

Let us consider now the deformation of the Kowalevski gyrostat proposed in [5, 2] with the following Hamiltonian

\[
H_d = J_1^2 + J_2^2 + 2J_3^2 + 2c_1x_1 - 2c_2J_3x_2 - c_2^2 x_3^2 + 2a_1 (J_3 + c_2x_3), \tag{3.2}
\]

(see equation (14) in [5]). In terms of the variables \( s_i \) and \( t_i \) this function reads

\[
H_d = (s, A_d s) + 2(s, B_d t) + (t, C_d t) + (c, s - t) + (d, s + t), \tag{3.3}
\]

where \( A_d \) and \( C_d \) are symmetric matrices

\[
A_d = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -\varkappa c_2 \\
-\varkappa c_2 & 2 - \varkappa^2 c_2^2
\end{pmatrix}, \quad C_d = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \varkappa c_2 \\
0 & \varkappa c_2 & 2 - \varkappa^2 c_2^2
\end{pmatrix}, \tag{3.4}
\]

whereas \( B_d \) is a skew-symmetric matrix

\[
B_d = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \varkappa c_2 \\
-\varkappa c_2 & 2 + \varkappa^2 c_2^2
\end{pmatrix},
\]

which does not commute with \( A \) and \( B \). The vectors \( c \) and \( d \) with numerical components are

\[
c = (2c_1 \varkappa^{-1}, 2a_1 c_2 \varkappa^{-1}, 0), \quad d = (0, 0, 2a_1).
\]

By use of the group elements of \( SO(3) \times SO(3) \) we can simultaneously bring the matrices \( A_d \) and \( C_d \) to a diagonal form. However, in this representation the third matrix \( B_d \) is not diagonal. Namely, after suitable transformation one gets

\[
A_d' = C_d' = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - \varkappa^2 c_2^2 & 0 \\
0 & 0 & 2
\end{pmatrix}, \quad B_d' = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - \varkappa^2 c_2^2 & -2\varkappa c_2 \\
0 & 2\varkappa c_2 & 2
\end{pmatrix}
\]

and

\[
c' = (2c_1 \varkappa^{-1}, 0, 0), \quad d' = (0, 0, 2a_1 \sqrt{1 + \varkappa^2 c_2^2}).
\]
It is an automorphism of the algebra $so(4)$, which transforms the Hamilton function $H_d$ (3.2) to the form
\[
H_d = J_1^2 + (1 - x^2 c_2^2)J_2^2 + 2J_3^2 + 2c_1 x_1 + 2c_2 (x_2 J_3 - x_3 J_2) + 2a_1 \beta^{-1} J_3.
\] (3.5)
The matrices $B_d$ and $B'_d$ are skew-symmetric in contrast to the matrices $B$ and $B'$. Hence, we can suppose that the Hamiltonians $H$ (2.7) and $H_d$ (3.2) describe different integrable systems.

In terms of the variables $J$ and $x$ the Hamilton function $H$ (2.7) is a linear-quadratic form in momenta $J$ and linear in the coordinates $x$
\[
H = (J, A) = (\tilde{a} J + (\tilde{a}, z) + 2(\tilde{b}, J)
\] (3.6)
where
\[
z = (p_1 + p_2)J + \chi^{-1}(p_1 - p_2) x - \chi^{-1}(x \times J).
\]
Here $(x \times J)$ stands for vector cross product and the symmetric matrix $\tilde{A}$ reads
\[
\tilde{A} = \begin{pmatrix}
2b_1b_0 & b_0c_1 + c_0b_1 & -d_1b_1 - a_1b_0 \\
b_0c_1 + c_0b_1 & 2c_1c_0 & -c_0a_1 - c_1d_1 \\
-d_1b_1 - a_1b_0 & -c_0a_1 - c_1d_1 & 2a_1d_1
\end{pmatrix}.
\]
The matrices $\tilde{a}$ and $\tilde{b}$ with numerical components have the form
\[
\tilde{a} = (c_1 d_1 - c_0 a_1, a_1 b_0 - d_1 b_1, b_0 c_1 - c_0 b_1), \quad \tilde{b} = (b_0 a_0 + d_0 b_1, a_0 c_0 + d_0 c_1, -a_1 d_0 - a_0 d_1).
\]
The second integral of motion
\[
\tilde{K} = K + \frac{(c_1 c_0 + a_1 d_1 + b_1 b_0)}{8\chi^2} c_2 \chi
\]
is a third order polynomial in the momenta $J$. Recall that the additional integral to the Hamiltonian $H_d$ (3.2) is a quartic polynomial in momenta $[5, 2]$.

4. Special case of the Poincaré system

In this section we prove that the dynamical system (2.7) may be considered as an integrable deformation of the known Poincaré system. Let
\[
c_1 = -\frac{a_0 c_0}{d_0}, \quad b_1 = -\frac{a_0 b_0}{d_0}, \quad a_1 = -\frac{a_0 d_1}{d_0},
\] (4.1)
so that $a = 0$ and $b = 0$ (2.8).

In this case, the Hamilton function (2.7) becomes a quadratic form in the variables $s$ and $t$. Moreover, the constraint (4.1) results in commutativity of the matrices $A$ and $B$. It means that we can bring both matrices $A$ and $B$ to a diagonal form simultaneously, and after a suitable rotation, the corresponding Hamilton function
\[
H_p = (s_1 + t_1)^2
\]
describes a special case of the Poincaré system $[3]$. The second integral of motion $K_p$ reads
\[
K_p = 2(b_0^2 + c_0^2 + d_0^2)\left[ (s \times t)_{1}^2 - (s, t)^2 \right] + 4d_0^2(s, t)
\]
\[
+ 4d_0 \sqrt{b_0^2 + c_0^2 + d_0^2} \left[ \left[ (s, t) + t^2 \right] s_1 - [s^2 + (s, t)] t_1 \right].
\]
Thus, we can consider system (2.7) as an integrable deformation of the Poincaré system with quartic additional integral of motion.
5. Summary

Using the standard $r$-matrix technique and the known representation of the Sklyanin algebra on $so(4)$, an integrable system with quadratic and quartic integrals of motion has been obtained. Furthermore, this technique also gives a quantum counterpart of the system.

The proposed integrable system requires special detailed treatment and may be considered as an integrable deformation of the known Poincaré system.

References