The method introduced in [20] was applied in [21] and [22] for constructing integrable conservative two-dimensional mechanical systems whose second integral of motion is polynomial up to third degree in the velocities. In this paper we apply the same method for systematic construction of mechanical systems with a quartic integral. As in our previous works, the configuration space is not assumed an Euclidean plane. This widens the range of applicability of the results to diverse mechanical systems such as rigid body dynamics and motion on two-dimensional surfaces of positive, negative and variable curvature. Two new several-parameter integrable systems are obtained, which unify and generalize several previously known ones by modifying the configuration manifold and the potential of the forces acting on the system. Those systems are shown to include as special cases, integrable problems of motion in the Euclidean plane, the hyperbolic plane and different types of curved two-dimensional manifolds. The results are applied to problems of rigid body dynamics. New integrable cases are obtained as special versions of one of the new systems, corresponding to different choices of the parameters. Those cases include new generalizations of the classical cases of Kovalevskaya, Chaplygin and Goryachev.

1. Introduction

Kovalevskaya's integrable case of the dynamics of a heavy rigid body moving about a fixed point was probably the first known case of a mechanical system having an integral quartic in velocities in addition to the energy integral [1]. It was followed shortly by the case due to Chaplygin of motion of a body in a liquid [2] (see also [25]). Up to now, only a very limited number of integrable cases of motion of a particle in the Euclidean plane with a quartic integral was found, mostly in the past twenty years or so (e.g. [4]–[16]). Most of those cases are listed in Hietarinta's review [3].

None of previous works was devoted to systematic search of polynomial integrals in cases when the configuration manifold is not an Euclidean plane. The two cases of Kovalevskaya and Chaplygin has remained until recently the only known examples of natural systems with a quartic integral on a two-dimensional curved manifold.

In virtue of the famous Maupertuis principle, the motion of a natural mechanical system can be brought into equivalence (more precisely, orbital equivalence) with the geodesic flow on some Riemannian metric. Metrics on the Riemannian sphere associated with known integrable cases of rigid body dynamics were constructed in [17]. Two families of integrable systems with a quartic integral on $S^2$ were obtained in [18] and [19].

The method introduced in our work [20] has proved successful in constructing several new many-parameter families of integrable two-dimensional (not necessarily plane) mechanical systems with integrals quadratic [21] and cubic [22] in velocities. Some of those systems unify and generalize certain previously known ones. In particular, the famous integrable cases of rigid body dynamics are all recovered and mostly generalized by introducing additional parameters into their structure.
The present paper is devoted to application of this method to construction of integrable systems with a complementary integral of the fourth degree in the velocities. The metric of the configuration space is determined in the process of solution, so that it may result in curved as well as flat two dimensional manifolds.

The paper is organized as follows: In the introduction the necessary preliminary considerations and review of the method used. Some points concerning this method can be found in more detail in our previous articles, e.g. such as the proof of reducibility of the form of the polynomial integral to the simple form involving only one term of the highest power. We also add some modification that allows systematically adding certain extra parameters to the structure of the system, depending on the structure of the potential function of that system, through a change of the independent variable (the time). Section 2 contains the main results. Two systems admitting quartic integral are constructed depending on 12 and 14 parameters, respectively. Some known and unknown integrable and super-integrable problems are pointed out as special cases of the two systems. In section 3 we tried to find all cases, when the 14-parameter system coincides through a point transformation with the system describing the motion of a rigid body with one point fixed in a potential field.

1.1. Reduction of the arbitrary natural system to the simplest form

Consider the natural conservative mechanical system described by the Lagrangian

$$L = \frac{1}{2} \sum_{i,j=1}^{2} a_{ij} \dot{q}_i \dot{q}_j - V$$

(1.1)

where $a_{ij}$, $V$ are certain functions of the generalized coordinates $q_1$, $q_2$ only. This system clearly admits the energy integral

$$I_1 = \frac{1}{2} \sum_{i,j=1}^{2} a_{ij} \dot{q}_i \dot{q}_j + V = h$$

(1.2)

with $h$ as the arbitrary energy parameter. It is evident that this system is time-reversible. This means that if an integral of motion polynomial in the velocities $\dot{q}_1$, $\dot{q}_2$ contains even and odd powers of the velocity variables, then the even and odd parts of this integral are both integrals of motion. As we are interested here in systems admitting a quartic integral, the most general form of this integral can be written as

$$I = \sum_{i=0}^{4} C_{4i} \dot{q}_1 \dot{q}_2^{4-i} + \sum_{i=0}^{2} C_{2i} \dot{q}_1 \dot{q}_2^{2-i} + C_0$$

(1.3)

where $C_{i,j}$ and $C_0$ are functions in $q_1$, $q_2$.

It is always possible to refer the system to isometric coordinates $x$, $y$ (say) on the configuration space. We can write the Lagrangian of this system as

$$L = \frac{1}{2} \Lambda (\dot{x}^2 + \dot{y}^2) - V$$

(1.4)

where $\Lambda$, $V$ are certain functions of $x$, $y$. The energy integral takes the form

$$I_1 = \frac{1}{2} \Lambda (\dot{x}^2 + \dot{y}^2) + V = h.$$  

(1.5)

1.2. Change of the independent variable

It is well known (e.g. Birkhoff [23] or [24]) that transforming time $t$ to a new independent variable $t_1$ by the relation

$$dt = \Lambda dt_1$$

(1.6)
the original system is transformed to the one with the Lagrangian

\[ L_1 = \frac{1}{2} \left[ x^2 + y^2 \right] - V_1 \]  

(1.7)

where the asterisk denotes derivative with respect to \( t_1 \). This system describes motion of a fictitious particle in the plane under the action of forces with potential \( V_1 = \Lambda(V - h) \) in which the energy constant of the original system already enters as a parameter. In order not to allow another energy constant from appearing during integration of the equations of motion of the new system, we restrict the last to its zero energy level, i.e. the free system (1.4) is equivalent to the system (1.7) under the restriction

\[ H_1 = \frac{1}{2} \left[ x^2 + y^2 \right] + V_1 = 0. \]  

(1.8)

The general quartic integral (1.3) is now expressed as the sum of three homogeneous polynomials in \( x, y \) all involving 9 coefficients as functions of the coordinates:

\[ I = \sum_{i=0}^{4} A_{4,i}^\prime x^{4-i} y^i + \sum_{i=0}^{2} A_{2,i}^\prime x^{2-i} y^{2-i} + A_0. \]  

(1.9)

1.3. Invariance under conformal mapping of the plane

After affecting an arbitrary conformal mapping

\[ x + iy = z(\zeta), \zeta = \xi + i\eta \]  

(1.10)

the Lagrangian (1.7) changes to

\[ L_1 = \frac{1}{2} \left[ \frac{dx}{d\zeta} \right]^2 \left[ \frac{d\eta}{d\zeta} \right]^2 - V_1. \]  

(1.11)

Applying the independent variable change

\[ dt_1 = \left| \frac{dz}{d\zeta} \right|^2 d\tau \]  

(1.12)

one can again reduce the Lagrangian to the particle form

\[ L_2 = \frac{1}{2} \left[ \xi'^2 + \eta'^2 \right] + U, \quad U = -\left| \frac{dz}{d\zeta} \right|^2 V_1 \]  

(1.13)

in which the primes denote derivatives with respect to \( \tau \). The corresponding equations of motion are

\[ \xi'' = \frac{\partial U}{\partial \xi}, \quad \eta'' = \frac{\partial U}{\partial \eta}, \]  

\[ \xi'^2 + \eta'^2 = 2U. \]  

(1.14) \hspace{1cm} (1.15)

1.4. Simplification of the form of the integral

As has been proved in [20] for the general case of a polynomial integral, using the energy integral (1.15) and a suitable conformal mapping in the transformation (1.10) we can always reduce the integral to the form

\[ I = \xi^4 + P\xi'^2 + Q\xi' \eta' + R = \text{const} \]  

(1.16)

involving only three functions of \( \xi, \eta \).
Let us now consider the system (1.14) restricted by the condition (1.15). If this system admits an integral of the form (1.16), the four unknown functions involved should satisfy the system of equations

\[ \frac{\partial P}{\partial \xi} - \frac{\partial Q}{\partial \eta} + 4 \frac{\partial U}{\partial \xi} = 0, \quad \frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial \xi} = 0 \]  
\[ \frac{\partial R}{\partial \xi} + 2P \frac{\partial U}{\partial \xi} + Q \frac{\partial U}{\partial \eta} + 2U \frac{\partial Q}{\partial \eta} = 0, \quad \frac{\partial R}{\partial \eta} + Q \frac{\partial U}{\partial \xi} = 0. \]  

(1.17)

(1.18)

From (1.18) one can express the function \( R \) up to an additive constant- in the form

\[ R(\xi, \eta) = - \int Q \frac{\partial U}{\partial \xi} d\eta - \int \left[ 2P \frac{\partial U}{\partial \xi} + Q \frac{\partial U}{\partial \eta} + 2U \frac{\partial Q}{\partial \eta} \right] d\xi \]  

(1.19)

where \([ \cdot ]_0\) means that the expression in the bracket is computed for \( \eta \) taking an arbitrary constant value \( \eta_0 \) (say). The whole system (1.17-1.18) is thus maximally reduced to the form of three equations

\[ \frac{\partial P}{\partial \xi} - \frac{\partial Q}{\partial \eta} + 4 \frac{\partial U}{\partial \xi} = 0, \quad \frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial \xi} = 0, \]  
\[ \frac{\partial}{\partial \eta} \left( 2P \frac{\partial U}{\partial \xi} + Q \frac{\partial U}{\partial \eta} + 2U \frac{\partial Q}{\partial \eta} \right) - \frac{\partial}{\partial \xi} \left( Q \frac{\partial U}{\partial \xi} \right) = 0. \]  

(1.20)

The set of solutions of the last system of equations generates all systems of the type (1.14) having an integral of the form (1.16) on the zero level of their energy integral (1.15). Affecting all possible transformations of types inverse to those of 1.3 and 1.2 followed by a general point transformation to the generalized coordinates \( q_1, q_2 \) we obtain all systems of the general form having a quartic integral on the zero level of their energy integral.

1.5. Classes of integrable systems on arbitrary energy level

All what we obtain by solving the system (1.20) is a Lagrangian of the form

\[ L = \frac{1}{2} (\xi^2 + \eta^2) + U, \]  

(1.21)

which admits a quartic integral on the zero-level of the integral

\[ \frac{1}{2} (\xi^2 + \eta^2) - U = 0. \]  

(1.22)

It may happen that in the process of solution we get a function \( U = h - V_0(\xi, \eta) \) involving an additive arbitrary constant \( h \). We rewrite (1.21), (1.22) in the form

\[ L = \frac{1}{2} (\xi^2 + \eta^2) + h - V_0, \]  

(1.23)

\[ \frac{1}{2} (\xi^2 + \eta^2) + V_0 = h. \]  

(1.24)

In the Lagrangian (1.23) the constant \( h \) is insignificant and can be cancelled out, while (1.24) shows that our system is valid on an arbitrary energy level \( h \). Here the independent variable \( \tau \) is interpreted as the natural time \( t \). The parameter \( h \) may enter in the coefficients for the integral. It may be treated either as signifying its numerical value (the arbitrary energy of the motion) or signifying its expression from (1.24) in terms of the coordinate and velocity variables.
Suppose that the function $U$ in (1.21) has the structure

$$U = A_1 v_1 + A_2 v_2 + \cdots - V_0$$

(1.25)

involving a number of free parameters $A_i, i = 1, 2, \cdots$ which enter only as linear multipliers and do not occur anywhere else in the Lagrangian\(^1\), the functions $v_i(\xi, \eta), i = 1, 2, \cdots$ being independent of those parameters. If $U$ has a free additive parameter, one can choose one of the functions $v$ say $v_1 = 1$ and denote that parameter by $A_1$. Suppose also that the system (1.21) admits a quartic integral subject to the restriction (1.22). We show now a method for transforming the Lagrangian to another one involving an additive arbitrary constant, which can be interpreted as a free energy constant. First, we introduce new free parameters $h, a_1, a_2, a_3, \cdots$ by the relations

$$A_i = a_i h$$

(1.26)

This gives $U$ the form

$$U = (a_1 v_1 + a_2 v_2 + \cdots) h - (a_1 v_1 + a_2 v_2 + \cdots + V_0).$$

(1.27)

Noting that a change of the independent variable according to the rule

$$d\tau = \frac{dt}{\chi}$$

(1.28)

in which $\chi = \chi(\xi, \eta)$, reduces the Lagrangian (1.21) to the form

$$L = \frac{1}{2} \chi \left( \dot{\xi}^2 + \dot{\eta}^2 \right) + \frac{1}{\chi} U$$

(1.29)

and choosing

$$\chi = a_1 v_1 + a_2 v_2 + \cdots$$

(1.30)

we get the final form of the Lagrangian:

$$L = \frac{1}{2} (a_1 v_1 + a_2 v_2 + \cdots) \left( \dot{\xi}^2 + \dot{\eta}^2 \right) + h - \frac{a_1 v_1 + a_2 v_2 + \cdots + V_0}{a_1 v_1 + a_2 v_2 + \cdots}$$

(1.31)

containing the additive arbitrary constant $h$, whose presence can be discarded. On the other hand, the restriction (1.22) is replaced by

$$\frac{1}{2} (a_1 v_1 + a_2 v_2 + \cdots) \left( \dot{\xi}^2 + \dot{\eta}^2 \right) + \frac{a_1 v_1 + a_2 v_2 + \cdots + V_0}{a_1 v_1 + a_2 v_2 + \cdots} = h$$

(1.32)

which indicates that the free parameter $h$ is the energy of the system given by (1.31) and $t$ is the natural time. On the mean time, the quartic integral (1.16) after the change (1.28) to natural time takes the form

$$I = \chi^4 \dot{\xi}^4 + \chi^2 \left( P \ddot{\xi}^2 + Q \dot{\xi} \dot{\eta} \right) + R = \text{const.}$$

(1.33)

It may happen that the coefficients $P, Q$ and $R$ depend on $\{A_i\}$ and hence on $h$. This parameter should be replaced wherever it occurs by its expression from (1.32). Note also that the arbitrary parameters $a_i$ remain as multipliers in the potential terms, while $\{a_i\}$ enter into the metric on the configuration space and may greatly widen the range of possible application of the results. Although splitting the constants $\{A_i\}$ into parts between potential and metric may seem artificial, we will see below some cases of real applications obtained namely in this way.

Thus we have constructed an unconditional integrable system from the conditional one corresponding to the structure (1.25). This construction will be used frequently in the rest of this paper.

\(^1\)In practice, the metric on the configuration space (equivalently the kinetic energy of the mechanical system) is frequently expressed in terms of other local variables and certain parameters. Such parameters cannot be taken into account even if they enter in $U$ in the way shown in (1.25).
2. Two new systems with quartic integral

In the present section we shall construct solutions of this system compatible with the assumption that the function \( U \) has the structure

\[
U = u(\eta) + v(\eta)\Phi(\xi). \tag{2.1}
\]

This choice is motivated by the most remarkable integrable systems with a quartic integral, Ko-valievskaya’s and Chaplygin’s cases of rigid body dynamics and their generalizations by Goryachev, whose potentials have the structure (2.1) after reduction to two-dimensional systems using the cyclic integral. It turned out that only two choices of the function \( \Phi(\xi) \) are possible, which we discuss in the next two sections. In fact, substituting (2.1) into the first equations in (1.20) we conclude that \( P, Q \) must be expressible in the form

\[
P = p_0 + p\Phi(\xi), \quad Q = q_0 + q\Phi'(\xi), \quad v = \frac{1}{4}(q' - p) \tag{2.2}
\]

where \( p, q, p_0 \) are functions of \( \eta \) and \( q_0 \) is a constant. Using (2.1), (2.2) in the second equation we get

\[
p_0' + p\Phi'(\xi) + q\Phi''(\xi) = 0. \tag{2.3}
\]

Differentiating this equation once with respect to \( \xi \) we arrive at the separable equation

\[
\frac{\Phi''''(\xi)}{\Phi'(\xi)} = -\frac{p'}{q} = -\mu \quad \text{(separation constant)} \tag{2.4}
\]

whose solution depends greatly on whether \( \mu = 0 \) or \( \mu \neq 0 \). Those two cases will be treated separately in the next two subsections:

2.1. The first system

2.1.1. The generic system

When \( \mu = 0 \) one obtains from (2.4) and (2.3)

\[
\Phi(\xi) = a\xi^2 + b\xi + c, \quad p = \text{const} = n, \quad q = -\frac{p_0}{a}. \tag{2.5}
\]

Inserting those expressions into (2.2) and then in the system (1.20) we find the solution of the last in the form

\[
P = p_0(\eta) + n\Phi(\xi), \quad Q = \frac{1}{2a}p_0'(\eta)\Phi'(\xi), \quad v = -\left[\frac{n}{4} + \frac{p_0''(\eta)}{8a}\right], \quad u = -\frac{n\alpha p_0(\eta)^2 + a_1 p_0(\eta) + a_0}{p_0'(\eta)^2} - \frac{p_0(\eta)}{4} \tag{2.6}
\]

involving the arbitrary constants \( n, a_0, a_1, c \) and a single function \( p_0(\eta) \) which satisfies the equation

\[
p_0'(\eta)p_0^{(4)}(\eta) + 5p_0''(\eta)p_0''(\eta) = 0. \tag{2.7}
\]

This equation can be solved in two different ways, leading to different parametrization of the solution. One of them is the following:
Let us rewrite (2.7) in the form
\[ \frac{d}{d\eta} \left( p_0'(\eta) p_0''(\eta) \right) + 4 p_0''(\eta) p_0'''(\eta) = 0 \]
and integrate once to get
\[ p_0'(\eta) p_0''(\eta) + 2 p_0''(\eta)^2 = \frac{c_2}{2}. \]
Multiplying on both sides by \( p_0'(\eta) \) and integrating again we obtain
\[ p_0'(\eta)^2 p_0''(\eta) = \frac{c_2}{2} p_0(\eta) + \frac{c_1}{4}. \]
Multiplying again by \( p_0'(\eta) \) and integrating we arrive at the separable equation
\[ p_0'^4 = c_2 p_0^2 + c_1 p_0 + c_0 \quad (2.8) \]
where \( c_0, c_1, c_2 \) are integration constants and in the last formula we drop the argument in the function \( p_0 \). We finally write down the general solution of (2.7) as
\[ \eta = \int \frac{d p_0}{\left[ c_2 p_0^2 + c_1 p_0 + c_0 \right]^{1/4}}. \quad (2.9) \]
The inversion of the last integral is not a single valued function in \( \eta \) for generic values of the parameters. It is thus more convenient to use \( p_0 \) instead of \( \eta \) as a variable. We first note that the term \( c \) in the expression (2.5) for \( \Phi \) can be discarded, since it adds to \( U \) in (2.1) a term \( cv(\eta) \) which can be attributed to the function \( u(\eta) \). The neat contribution of this term will be an absolute constant, \( K \) (say) to the Lagrangian \( L \), at which we arrive after performing necessary manipulations in (2.6) and (2.1) – (2.2):
\[
L = \frac{1}{2} \left( \xi^2 + \frac{p_0^2}{\sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} \right) - \frac{nap_0^2 + a_1 p_0 + a_0}{\sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} - \frac{p_0}{4} - \left[ \frac{n}{4} + \frac{2c_2 p_0 + c_1}{32a\sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} \right] \xi(a \xi + b) + K. \quad (2.10)
\]
The corresponding quartic integral has the form
\[
I = \dot{\xi}^4 + [p_0 + n(\alpha \xi + b) - 4K] \dot{\xi}^2 - \frac{2a}{2a} \dot{\xi} p_0 - \frac{b}{2a} \left( 4nap_0 + \sqrt{c_2 p_0^2 + c_1 p_0 + c_0} \right) - \frac{1}{8a} \left[ 4 nap_0 + 16 na K + 4a_1 + \sqrt{c_2 p_0^2 + c_1 p_0 + c_0} \right] \xi(a \xi + b) + \frac{(16a^2 - c_2)}{64a^2} \dot{\xi}^2 (a \xi + b)^2. \quad (2.11)
\]
The Lagrangian (2.10) involves 9 parameters \( c_0, c_1, c_2, a, b, n, a_0, a_1, K \) of which the additive constant \( K \) can be considered as the energy constant. It can be ignored in the Lagrangian, with no effect on the equations of motion, and substituted in the integral (2.11) by its expression
\[
I_1 = \frac{1}{2} \left( \xi^2 + \frac{p_0^2}{\sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} \right) + \frac{nap_0^2 + a_1 p_0 + a_0}{\sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} + \frac{p_0}{4} + \left[ \frac{n}{4} + \frac{2c_2 p_0 + c_1}{32a\sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} \right] \xi(a \xi + b) = K.
\]
in terms of the state variables to obtain the uncoditional integral of the system.
2.1.2. Generalization through time transformation

We now apply the considerations of section 1.5 to introduce a much wider generalization of this system in the following way. We define new constants by the relations

\[
\begin{align*}
    a_0 &= h_0 + \alpha_0 h \\
    a_1 &= h_1 + \alpha_1 h \\
    K &= h_2 - \alpha_2 h
\end{align*}
\] (2.12)

and perform the time transformation

\[
dt \to \frac{dt}{G} \quad \text{where} \quad G = \frac{\alpha_1 p_0 + \alpha_0}{\sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} + \alpha_2.
\] (2.13)

Thus we obtain the Lagrangian

\[
L = \frac{1}{2} \left[ \left( \frac{\alpha_1 p_0 + \alpha_0}{\sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} + \alpha_2 \right) \dot{\xi}^2 + \left( \frac{\alpha_1 p_0 + \alpha_0 + \alpha_2 \sqrt{c_2 p_0^2 + c_1 p_0 + c_0}}{c_2 p_0^2 + c_1 p_0 + c_0} \right) p_0^2 \right] - \\
- \frac{h_0 + h_1 p_0 + n a p_0^2 + \left( \frac{1}{4} p_0 - h_2 \right) \sqrt{c_2 p_0^2 + c_1 p_0 + c_0}}{\alpha_1 p_0 + \alpha_0 + \alpha_2 \sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} - \\
- \frac{2 c_2 p_0 + c_1 + 8 a n \sqrt{c_2 p_0^2 + c_1 p_0 + c_0}}{32 a \left[ \alpha_1 p_0 + \alpha_0 + \alpha_2 \sqrt{c_2 p_0^2 + c_1 p_0 + c_0} \right]} \xi (a \xi + b) + h
\] (2.14)

in which h is the free energy constant and six parameters enter in the line element. This Lagrangian is integrable on arbitrary energy level \(h\). Its energy integral is

\[
I_1 = \frac{1}{2} \left[ \left( \frac{\alpha_1 p_0 + \alpha_0}{\sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} + \alpha_2 \right) \dot{\xi}^2 + \left( \frac{\alpha_1 p_0 + \alpha_0 + \alpha_2 \sqrt{c_2 p_0^2 + c_1 p_0 + c_0}}{c_2 p_0^2 + c_1 p_0 + c_0} \right) p_0^2 \right] + \\
+ \frac{h_0 + h_1 p_0 + n a p_0^2 + \left( \frac{1}{4} p_0 - h_2 \right) \sqrt{c_2 p_0^2 + c_1 p_0 + c_0}}{\alpha_1 p_0 + \alpha_0 + \alpha_2 \sqrt{c_2 p_0^2 + c_1 p_0 + c_0}} + \\
+ \frac{2 c_2 p_0 + c_1 + 8 a n \sqrt{c_2 p_0^2 + c_1 p_0 + c_0}}{32 a \left[ \alpha_1 p_0 + \alpha_0 + \alpha_2 \sqrt{c_2 p_0^2 + c_1 p_0 + c_0} \right]} \xi (a \xi + b) = h.
\] (2.15)

The corresponding quartic integral can be obtained from (2.11) by the transformation

\[
I \left( \xi, p_0, \dot{\xi}, \dot{p}_0 \right) \to I \left( \xi, p_0, G \dot{\xi}, G \dot{p}_0 \right)
\]. (2.16)

This gives
2.1.3. Special cases:

The gaussian curvature of the configuration manifold of the system with Lagrangian (2.14) can be calculated as

\[
\kappa = -\frac{1}{2G} \left( c_2 p_0^2 + c_1 p_0 + c_0 \right) \frac{1}{\left( \frac{d}{dp}_0 \right)} \ln(G). \tag{2.18}
\]

**Cases of flat configuration space**

1. The original family of systems characterized by the Lagrangian (2.10) in the \( \xi \eta \)-plane. This family corresponds to the choice \( \alpha_1 = \alpha_0 = 0, \alpha_2 = 1 \). In this case the potential is written in terms of the auxiliary variable \( p_0 \), which cannot be expressed explicitly in general in terms of \( \eta \) through the inversion of the integral (2.9). However, two cases are known to be expressed in Cartesian coordinates:

a) When in (2.10) we set \( c_2 = 0 \), we arrive after some change of variables at the problem of motion of a particle in the Euclidean \( xy \)-plane under the action of forces with potential

\[
V = 9a(a_0 y^2 + a_1 y_4^{1/3}) + a_2 y_4^{2/3} + a_3 y_4^{2/3} + 2x(ax + b)(2a_0 + a_1 y_4^{-2/3}). \tag{2.19}
\]

This coincides with the Holt-type potential introduced in [6] with the corresponding quartic integral given in [3].

b) When the polynomial under the quadratic root in (2.10) becomes a complete square and then, without loss of generality, one can set \( c_0 = c_1 = 0 \). This leads to the case of motion in the Euclidean plane with the potential

\[
V = a(x^2 + y^2) + bx + \frac{c}{y^2}(-hp) \tag{2.20}
\]

where we leave an additive constant \( h_p \) for future need to play the role of the system’s energy. This is a superintegable case, separable in Cartesian, elliptic and polar coordinates [4], [7]. The quartic integral in this case should be composed of the three quadratic separation integrals.
2. Another case of zero curvature is obtained when \( c_1 = c_2 = 0 \). This leads to a case of motion in the Euclidean plane with the potential

\[
V = \frac{1}{2}a(x^2 + 4y^2) + bx.
\]  

(2.21)

This potential contains one parameter less than a superintegrable case pointed out by Karlovini et al [8].

**Configuration space of constant curvature** The condition that \( \frac{dc}{dp_0} \equiv 0 \) leads to the former two cases of zero curvature and to one more case characterized by the choice \( \alpha_1 = \alpha_2 = c_0 = c_1 = 0 \) and then, without reducing generality, one can put \( \alpha_0 = 1, c_2 = \frac{4}{R^2} \). This gives for the metric of the configuration space the constant negative curvature

\[
\kappa = -\frac{1}{R^2}
\]  

(2.22)

so that we deal with motion on the Lobachevsky hyperbolic plane (or on a pseudosphere of radius \( R \) locally isometric to that plane). The Lagrangian of this system can be written in the isometric coordinates \((x, y)\) in the form

\[
L = \frac{1}{2}R^2 (\dot{x}^2 + \dot{y}^2) - \frac{y^2}{R^2} [a(x^2 + y^2) + bx + c] \left( +h_{hp} \right)
\]  

(2.23)

\( h_{hp} \) is the energy of the system. Integrable systems on the hyperbolic plane or the pseudosphere are rarely treated in the literature, and mainly in the context of superintegrability (see e.g. [9] and [10]). An integrable system with complementary cubic in velocities integral was pointed out in [22]. It can be at once realized that the Lagrangian (2.23) is transformed by the time change \( dt \rightarrow \frac{R^2}{y^2} dt \) to

\[
\frac{1}{2} (\dot{x}^2 + \dot{y}^2) - [a(x^2 + y^2) + bx + c] + h_{hp} \frac{R^2}{y^2}.
\]  

(2.24)

That represents a plane system with the same potential as (2.20) only with the replacement \( c \rightarrow -h_{hp}, h_{hp} \rightarrow -\frac{c}{R^2} \). Therefore the system (2.23) is superintegrable.

**2.1.4. A transformed form of the previous system**

Equation (2.7) can be put in the form

\[
\frac{p_0^{(4)}(\eta)}{p_0''(\eta)} + 5 \frac{p_0''(\eta)}{p_0'(\eta)} = 0
\]  

(2.25)

and integrated to give

\[
p_0'(\eta) \frac{p_0''(\eta)}{p_0'(\eta)} = \text{const} = -2C_0.
\]  

(2.26)

Denoting \( p_0'(\eta) \) by \( z \) and integrating with respect to \( z \), we get

\[
z^2 = \frac{C_0}{z^4} + C_1
\]  

(2.27)

where \( C_1 \) is a new constant of. On separation of variables this gives

\[
\eta = \eta_0 + \int \frac{z^2 dz}{\sqrt{C_0 + C_1 z^4}}
\]  

(2.28)
and hence

$$p_0 = \int zd\eta = \frac{\sqrt{C_0 + C_1 z^2}}{2C_1} - 4K.$$  

(2.29)

This is equivalent to making the substitution

$$c_2 p_0^2 + c_1 p_0 + c_0 = z^4$$  

(2.30)

and renaming the constants. The Lagrangian (2.10) is transformed to the form

$$L = \frac{1}{2} \left( \xi^2 + \frac{z^4 \dot{z}^2}{C_0 + C_1 z^4} \right) - \left[ \frac{na}{4C_1} z^2 + \frac{A}{z^2} + \frac{\sqrt{C_0 + C_1 z^4}}{z^2} \left( \frac{1}{8C_1} + \frac{B}{z^2} \right) - \frac{1}{8} \left( 2n + \frac{\sqrt{C_0 + C_1 z^4}}{az^2} \right) \xi (a \xi + b) \right] + K$$  

(2.31)

and the integral (2.11) to

$$I = \dot{\xi}^4 + \frac{1}{2C_1} \left[ \sqrt{C_0 + C_1 z^4} - 8C_1 K + 2nC_1 \xi (a \xi + b) \right] \dot{\xi}^2 - \frac{1}{2a} \frac{z^3 (2a \xi + b)}{\sqrt{C_0 + C_1 z^4}} \dot{\xi}^2 - \frac{b^2}{32C_1 a^2} \left[ C_1 z^2 + 2n a \sqrt{C_0 + C_1 z^4} \right] - \frac{1}{8C_1 a} \left[ C_1 z^2 + 2n a \sqrt{C_0 + C_1 z^4} + 16 n a C_1 K + 8 B C_1^2 \right] \xi (a \xi + b) + \frac{1}{16a^2} (4n^2 a^2 - C_1) \dot{\xi}^2 (a \xi + b)^2.$$  

(2.32)

One can use the three parameters $K$, $A$, $B$ and one of the parameters $n$ or $b$ to construct the class of systems homotopic to the one described by the Lagrangian (2.31).

### 2.2. The second system

For this system, from (2.3) and (2.4) one must have $p_0 = \text{const}$ and $\Phi(\xi)$ should satisfy the equation

$$\Phi''(\xi) + \mu \Phi'(\xi) = 0$$  

(2.33)

where $\mu$ is a nonzero constant. Noting that an additive constant in $\Phi(\xi)$ gives rise terms of $U$ independent of $\xi$ and those terms can be absorbed into the function $u(\eta)$, one can write the solution of the last equation as

$$\Phi(\xi) = \begin{cases} a \cos(\sqrt{\mu} \xi) + b \sin(\sqrt{\mu} \xi) & \mu > 0 \\ a \exp(\sqrt{-\mu} \xi) + b \exp(-\sqrt{-\mu} \xi) & \mu < 0. \end{cases}$$  

(2.34)

In a way similar to that of the previous section, the Lagrangian and the integrals of the system in this case can be expressed in the form

$$L = \frac{1}{2} \left( \dot{\xi}^2 + \dot{\eta}^2 \right) + \frac{1}{4} \mu p_0 (p_0^2 - a_1 p) - \frac{a_2}{p_0^2} + \frac{[p''(\eta) - \mu p(\eta)] \Phi(\xi)}{4\mu}.$$  

(2.35)
\[ I = \dot{\xi}^4 + [p_0 + p(\eta)\Phi(\xi)]\dot{\xi}^2 + \frac{1}{\mu^2}p(\eta)\Phi'(\xi)\dot{\xi}\dot{\eta} + \frac{\Phi'(\xi)^2 + \mu\Phi(\xi)^2}{8\mu^2} \left[ \mu p(\eta)^2 - p'(\eta)^2 \right] + \]
\[ + \frac{a_1}{\mu^2} \Phi(\xi) - \frac{1}{8\mu^2} \left[ p'(\eta)p''(\eta) + 2p''(\eta)^2 - \mu p'(\eta)^2 - 2\mu^2 p(\eta)^2 \right] \Phi(\xi)^2 \]

(2.36)

where \( a_1, a_2 \) are integration constants and \( p \) satisfies the fourth order equation
\[ p'(\eta)p^{(4)}(\eta) + 5p''(\eta)p'''(\eta) - 6\mu^2 p(\eta)p'(\eta) = 0. \]

(2.37)

Special versions and particular solutions of (2.37) were used in [18] and [19] in process of constructing families of integrable systems on the sphere \( S^2 \).

We first note that (2.37) can be reduced to the first order fourth degree equation
\[ p'^4 = \mu^2 p^4 + c_2 p^2 + c_1 p + c_0 \]

(2.38)
in which \( c_2, c_1, c_0 \) are arbitrary integration constants. Hence \( p \) is obtained as a function of \( \eta \) by inverting the relation
\[ \eta = \int \frac{dp}{\sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} \]

(2.39)

Instead of that we shall use \( p \) as an independent variable, so as to obtain
\[ L = \frac{1}{2} \left( \dot{\xi}^2 + \frac{\dot{p}^2}{\sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} \right) + \frac{1}{4} \mu p_0 p^2 - a_1 p - a_2 - \frac{p_0}{4} + \]
\[ + \frac{4\mu^2 p^3}{16\mu \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} \Phi(\xi) \]

(2.40)

\[ I = \dot{\xi}^4 + (p\Phi(\xi) + p_0)\dot{\xi}^2 + \frac{\Phi'(\xi)\dot{\xi} \dot{p}}{\mu} + \]
\[ + \frac{a_1}{\mu^2} \Phi(\xi) - \frac{1}{16\mu^2} \left( 2\mu^2 p^2 + c_2 - 2\mu \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0} \right) \Phi(\xi)^2 + \]
\[ + \frac{\Phi'(\xi)^2 + \mu \Phi(\xi)^2}{8 \mu^2} \left( \mu p^2 - \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0} \right) . \]

(2.41)

The Lagrangian (2.40) contains no free additive parameters and thus it admits the integral (2.41) only on its zero energy level. We now note a very special case subject to the conditions \( \mu = -1, c_2 = = c_1 = c_0 = 0 \). In this case the Lagrangian contains an additive arbitrary constant \( -\frac{p_0}{2} \), which can be considered as the energy constant and the system becomes general integrable. Lagrangian and the integral can be written in the form
\[ L = \frac{1}{2} (x^2 + y^2) - (ae^{y+x} + be^{-y-x} + ce^{-y} + de^{-2y}) \]

(2.42)

and
\[ I = \dot{x}^4 + 2(-h + ae^{y+x} + be^{-y-x}) \dot{x}^2 - 2e^y (ae^x - be^{-x}) \dot{x} \dot{y} - \]
\[ - e^{2y} (a^2 e^{2x} + b^2 e^{-2x} - 2ab) - 2e (ae^x + be^{-x}) . \]

(2.43)

This is a Toda-type system but with all coefficients in the potential arbitrary rather than equal as in previous results (see e.g. [3]).
Noting again that the system (2.40) contains the parameters $a_1, a_2, p_0$ that enter linearly in the potential\[^2\] but not in the line element of the configuration space, we change those parameters by the substitution

$$a_1 = h_1 - \alpha_1 h, \quad a_2 = h_2 - \alpha_2 h, \quad p_0 = 4(h_3 + \alpha_2 h).$$

Then taking

$$\Lambda = \frac{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0}{\sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} - \alpha_2$$

and changing the time variable we affect the transformation of the Lagrangian

$$L(\xi, p, \xi, \dot{p}) \rightarrow \frac{1}{\lambda} L(\xi, p, \Lambda \xi, \Lambda \dot{p})$$

to obtain the new Lagrangian

$$L = \frac{1}{2} \left[ \left( \frac{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0}{\sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} - \alpha_2 \right) \dot{\xi}^2 + \right.$$

$$\left. + \left( \frac{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0 - \alpha_2 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}}{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0} \right) \dot{p}^2 + \right.$$

$$\left. + \frac{\mu h_3 p^2 - h_1 p - h_2}{\sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} - \frac{\mu h_3 p^2 + h_1 p + h_2 - h_3 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}}{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0 - \alpha_2 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} + \right.$$

$$\left. + \frac{4 \mu^2 p^3 + 2 c_2 p + c_1 - 4 \mu p \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}}{16 \mu} \left( \frac{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0 - \alpha_2 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}}{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0} \right) \Phi(\xi) \right] = h. \tag{2.47}$$

This Lagrangian contains a free additive parameter $h$, which we will consider as the energy constant. The system is general integrable (for arbitrary energy). The energy integral is

$$I_1 = \frac{1}{2} \left[ \left( \frac{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0}{\sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} - \alpha_2 \right) \dot{\xi}^2 + \right.$$

$$\left. + \left( \frac{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0 - \alpha_2 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}}{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0} \right) \dot{p}^2 + \right.$$

$$\left. - \left\{ \frac{\mu h_3 p^2 - h_1 p - h_2}{\sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} - \frac{\mu h_3 p^2 + h_1 p + h_2 - h_3 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}}{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0 - \alpha_2 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} + \right.$$

$$\left. + \frac{4 \mu^2 p^3 + 2 c_2 p + c_1 - 4 \mu p \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}}{16 \mu} \left( \frac{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0 - \alpha_2 \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}}{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0} \right) \Phi(\xi) \right\} = h. \tag{2.48}$$

\[^2\]The method applies equally well to the two parameters $a,b$ entering in the function $\Phi(\xi)$. For simplicity we take only parameters that lead to a metric depending on the single variable $\eta$. 

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Two-Dimensional Conservative Mechanical Systems
We also obtain the complementary quartic integral from (2.41) using the transformation

\[ I(\xi, p, \dot{\xi}, \dot{p}) \rightarrow I(\xi, p, \Delta \dot{\xi}, \Delta \dot{p}) \]

(2.49)

so that we have

\[ I = \left( \frac{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0}{\sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} - \alpha_2 \right)^4 \dot{\xi}^4 + \]

\[ + \left( \frac{\mu \alpha_2 p^2 + \alpha_1 p + \alpha_0}{\sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0}} - \alpha_2 \right)^2 \left[ (p \Phi(\xi) + 4 h_3 + 4 I_1 \alpha_2) \dot{\xi}^2 + \frac{\Phi'(\xi)}{\mu} \dot{\xi} \dot{p} \right] + \]

\[ + \frac{h_1 - \alpha_1 I_1}{\mu} \Phi(\xi) - \frac{1}{16 \mu^2} \left( 2 \mu^2 p^2 + c_2 - 2 \mu \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0} \right) \Phi(\xi)^2 + \]

\[ + \frac{\Phi'(\xi)^2 + \mu \Phi(\xi)^2}{8 \mu^2} \left( \mu p^2 - \sqrt{\mu^2 p^4 + c_2 p^2 + c_1 p + c_0} \right) \]

(2.50)

where \( I_1 \) can be replaced either by its expression (2.48) in terms of the state variables or by its value \( h \).

**2.2.1. Applications**

The last system (2.47) involves 14 parameters

\[ \alpha_2, \alpha_1, \alpha_0, \mu, c_2, c_1, c_0, a_1, a_2, h_1, h_2, h_3, a, b \]

of which seven parameters enter in the line element of the configuration space. To demonstrate the richness of this system we tried to isolate two interesting special cases: 1) when the configuration space becomes either a flat manifold or one with constant (positive or negative) curvature and 2) when it coincides with the configuration space of the reduced problem of motion of a solid about a fixed point in an axially symmetric potential field on the zero level of the cyclic constant. Those two cases are discussed in the rest of this paper.

**Plane motion of a particle** For the configuration space of the system described by the Lagrangian (2.47) to be interpreted as an Euclidean plane it is necessary that the gaussian curvature of the metric vanishes identically, that is

\[ \kappa = -\frac{1}{2 \Lambda} \left[ (\mu p^4 + c_2 p^2 + c_1 p + c_0) \frac{1}{\mu} \frac{d}{dp} \right]^2 \ln \Lambda \equiv 0 \]

(2.51)

where \( \Lambda \) is given by (2.45). This leads to the construction of two different systems. The Lagrangian for the first one can be written as

\[ L = \frac{1}{2} \left( r^2 + r^2 \dot{\theta}^2 \right) - \frac{A e^x + B e^{-x}}{r^3} - \frac{a}{r^2} - \frac{b}{r} \]

(2.52)

in which \( r, \theta \) are polar coordinates in the plane. In this interpretation the integrable Lagrangian (2.52) suffers a continuity problem since the potential is not periodic in the angular variable \( x \). However, affecting the transformation

\[ r \rightarrow e^y, \quad dt \rightarrow r^2 dt \]

(2.53)

we note that this system coincides with (2.42) of the previous section.

The second system is characterized by the choice

\[ \alpha_1 = 0, \quad \alpha_2 = -\frac{1}{2} c_2 \alpha_3, \quad c_0 = \frac{1}{4} c_2^2, \quad \mu = \pm 1. \]

(2.54)
After some manipulations we can reduce the Lagrangian of this system for \( \mu = -1 \) to
\[
L = \frac{1}{2} \left[ \dot{x}^2 + \dot{y}^2 + (Ae^x + Be^{-x})(Ce^y + De^{-y}) - \frac{b + a(Ce^y + De^{-y})}{(Ce^y - De^{-y})^2} \right]
\] (2.55)
and the quartic integral to
\[
I = 4\dot{x}^4 - 4 \left[ h + (Ae^x + Be^{-x}) (Ce^y + De^{-y}) \right] \dot{x}^2 + 4 \left[ (Ae^x - Be^{-x}) (Ce^y - De^{-y}) \right] \dot{x} \dot{y} + 4 \left( BD + ACe^y e^{2x} \right) (e^{-2y} AD + e^{-2x} CB) - (Ae^x + Be^{-x})^2 (Ce^y + De^{-y})^2 + 2a (Be^{-x} + Ae^x).
\] (2.56)

For \( \mu = 1 \) we obtain the Lagrangian
\[
L = \frac{1}{2} \left[ \dot{x}^2 + \dot{y}^2 - a \cos x \cos y - \frac{b \cos y + c}{\sin^2 y} \right]
\] (2.57)
which gives a special case of a result found by Bozis [11].

**Motion on sphere and hyperbolic plane** The requirement that \( \frac{d\xi}{dp} \equiv 0 \) leads to two sets of conditions which correspond to cases of motion on a manifold of constant (nonzero) curvature:

a) The first set is \( \alpha_1 = \alpha_2 = c_1 = 0, c_2^2 - 4\mu^2 c_0 = 0 \). The last condition implies that the expression under the square root in (2.47) is a complete square. If we assign the positive sign to the extracted root we get a Lagrangian that is independent of the variable \( \xi \). Assigning the negative sign gives the metric
\[
ds^2 = \alpha_0 \left( \frac{d\xi^2}{- \mu p^2 - \frac{c_2^2}{2\mu}} + \frac{dp^2}{(- \mu p^2 - \frac{c_2^2}{2\mu})^2} \right)
\] (2.58)
corresponding to the kinetic energy in (2.47). This metric has the gaussian curvature
\[
\kappa = \frac{c_2}{2\alpha_0}
\] (2.59)
which can be positive or negative according to the two parameters involved. However, that is not all yet, the metric and the kinetic energy should be positive definite. Regarding the last term in (2.58), it follows that \( \alpha_0 > 0 \), so that one may write \( \alpha_0 = 1 \). Two possibilities occur:

1) Either \( \mu > 0 \) and then \( c_2 \) must be negative and hence giving only negative curvature. Example is the case \( \mu = 1, c_2 = -2 \) which after a simple change of variables and renaming the parameters leads to the following Lagrangian with a quartic integral
\[
L = \frac{1}{2} \left( \cosh^2 y \dot{x}^2 + \dot{y}^2 \right) - \left[ \frac{\sinh y}{\cosh^2 y} \left( a \cos x + b \sin x \right) + k_1 + k_2 \sinh y \right]
\] (2.60)
which may be interpreted as describing motion on a pseudo-sphere.

2) or \( \mu < 0 \) and then \( c_2 \) may have any sign, giving a sphere for positive and a pseudo-sphere for negative \( c_2 \). We give, as example, the details for the case of a sphere. The Lagrangian in the usual spherical coordinates
\[
L = \frac{1}{2} \left( \sin^2 \theta \dot{\phi}^2 + \dot{\theta}^2 \right) - \left[ k_1 \cos \theta \frac{\cos \theta}{\sin \theta} - 2k_2 \sin^2 \theta + (ae^\phi + be^{-\phi}) \cos \theta \frac{\cos \theta}{\sin^3 \theta} \right]
\]
and the quartic integral
\[ I = \sin^8 \theta \varphi^4 + \left[ 2 \cos \theta (ae^x + be^{-x}) - 4k_2 \sin \theta \right] \sin^3 \theta \varphi^2 + 2 \sin^2 \theta (ae^x - be^{-x}) \varphi \theta - 2k_1 (ae^x + be^{-x}) - \frac{(ae^x - be^{-x})^2}{\sin^2 \theta}. \] (2.61)

Being non-periodic functions in the longitudinal variable \( \varphi \), the given integrable problem is of little use in real problems.

b) The conditions \( \alpha_2 = c_1 = 0, c_2^2 - 4\mu^2c_0 = 0, 2\mu^2\alpha_0^2 + c_2\alpha_1^2 = 0 \) give the curvature \( \kappa = -\frac{\mu^2\alpha_0}{2\alpha_1} \), whose sign is opposite to that of \( \alpha_0 \). Example of the motion on the sphere is
\[ L = 1 \left( \sin^2 \theta \varphi^2 + \theta^2 \right) - \left[ \frac{A}{\cos^2 \theta} + \frac{B}{\sin^2 \theta} + \frac{(ae^x + be^{-x})(2 - \sin^2 \theta)}{\sin^4 \theta} \right]. \] (2.62)

It has the same drawback as the last example.

3. Application to rigid body dynamics

3.1. Equations of motion of the rigid body

Consider the problem of motion of a rigid body about a fixed point under the action of a combination of conservative potential and gyroscopic forces, described by the Lagrangian:
\[ L = \frac{1}{2} \omega I \cdot \omega + 1.\omega - V, \] (3.1)

where \( I = diag(A,B,C) \) is the inertia matrix of the body. Assume that all the forces acting on the body have the Z-axis (say) as a common axis of symmetry. The potential \( V \) and the vector \( l \) depend only on the Eulerian angles \( \theta \) (of nutation) and \( \varphi \) (of proper rotation) through the direction cosines \( \gamma = (\gamma_1, \gamma_2, \gamma_3) = (\sin \theta \sin \varphi, \sin \theta \cos \varphi, \cos \theta) \) of the Z-axis. The equations of motion of a rigid body are usually written in the Euler - Poisson variables \( \omega, \gamma \). For the present problem this form, corresponding to (3.1), is:

\[ \dot{\omega} I + \omega \times (\omega I + \mu) = \gamma \times \frac{\partial V}{\partial \gamma}, \]
\[ \dot{\gamma} + \omega \times \gamma = 0 \] (3.2)

where
\[ \mu = \frac{\partial}{\partial \gamma} (1 \cdot \gamma) - \left( \frac{\partial}{\partial \gamma} \cdot 1 \right) \gamma. \] (3.3)

For such system the angle of precession \( \psi \) around the Z-axis is a cyclic variable. Moreover, we will restrict our consideration to the case the body exhibits axial dynamical symmetry \( A = B \) and the vector \( l \) lies along the axis of dynamical symmetry i.e. \( l = (0,0,l_3) \). The Lagrangian of the system will have the form
\[ L = \frac{1}{2} \left[ A(p^2 + q^2) + Cr^2 \right] + l_3r - V. \] (3.4)

It will be more convenient for our purpose to write this Lagrangian explicitly in terms of the Eulerian angles as generalized coordinates
\[ L' = \frac{1}{2} \left[ A(\dot{\psi}^2 + \sin^2 \theta \dot{\varphi}^2) + C(\dot{\psi} \cos \theta + \dot{\varphi})^2 \right] + l_3(\dot{\psi} \cos \theta + \dot{\varphi}) - V. \] (3.5)
The cyclic integral for (3.5) can be written as

\[ D\dot{\psi} + (C\dot{\phi} + l_3)\cos \theta = \text{const} = f \]  

where \( D = A\sin^2 \theta + C\cos^2 \theta \). That is

\[ \dot{\psi} = \frac{f - (C\dot{\phi} + l_3)\cos \theta}{D}. \]  

Ignoring \( \psi \) we construct the Routhian

\[ R = \frac{1}{2} \left( \dot{\gamma}_3^2 + \frac{C(1 - \gamma_3^2)p^2}{A - (A - C)\gamma_3^2} \right) + \frac{(fC\cos \theta + Al_3 \sin \theta)\dot{\phi}}{A[A - (A - C)\cos^2 \theta]} \]

\[ - \frac{1}{A} \left( V + \frac{(f - l_3 \cos \theta)^2}{2[A - (A - C)\cos^2 \theta]} \right) \]  

or in terms of \( \gamma_3 \)

\[ R = \frac{1}{2} \left( \dot{\gamma}_3^2 + \frac{C(1 - \gamma_3^2)p^2}{A - (A - C)\gamma_3^2} \right) + \frac{(fC\gamma_3 + Al_3(1 - \gamma_3^2))\dot{\phi}}{A[A - (A - C)\gamma_3^2]} \]

\[ - \frac{1}{A} \left( V + \frac{(f - l_3\gamma_3)^2}{2[A - (A - C)\gamma_3^2]} \right). \]  

### 3.2. New integrable cases of motion of the rigid body

As long as we study time-reversible mechanical systems we have to add on the above problem the restrictions that \( l_3 = 0, f = 0 \), so that the Routhian becomes

\[ R = \frac{1}{2} \left( \frac{\dot{\gamma}_3^2}{1 - \gamma_3^2} + \frac{C(1 - \gamma_3^2)p^2}{A - (A - C)\gamma_3^2} \right) - \frac{1}{A} V. \]  

The question of existence of a quartic integral of motion of the rigid body acted upon by forces with potential

\[ V = V_0(\gamma_3) + V_1(\gamma_3)\sin(\omega\varphi) \]

is solved by finding all possible cases when the metric part of (2.47) coincides (for \( \mu = \omega^2 \)) with the same part of (3.10). This is equivalent to the condition that the transformation involving the variables \( \gamma_3, p \)

\[ \frac{\alpha_2p^2 + \alpha_1p + \alpha_0}{\sqrt{\mu p^4 + c_2 p^2 + c_1 p + c_0}} - \alpha_2 = \frac{C(1 - \gamma_3^2)}{A - (A - C)\gamma_3^2} \]  

implies that

\[ \frac{\alpha_2p^2 + \alpha_1p + \alpha_0 - \alpha_2\sqrt{\mu p^4 + c_2 p^2 + c_1 p + c_0}}{\mu p^4 + c_2 p^2 + c_1 p + c_0} = \frac{1}{1 - \gamma_3^2} \left( \frac{d\gamma_3}{dp} \right)^2. \]  

Solving (3.11) in \( \gamma_3 \) and substituting in (3.12) and requiring the resulting relation to be an identity in the variable \( p \), we obtained a system of 22 polynomial equations in the parameters of the problem. The following four cases valid under the conditions

\[ A = 2C, \quad \alpha_2 = -\frac{1}{2} \]  

seem to exhaust all acceptable solutions of (3.11) – (3.12):
1. For
\[ \mu = 1, \quad c_2 = 1, \quad \alpha_0 = \alpha_1 = c_0 = c_1 = 0 \quad \text{and} \quad p = -\frac{\cos^2 \theta}{\sin \theta} \] (3.14)
we obtain the integrable problem with potential
\[ V = a \sin \theta \sin(\varphi - \varphi_0) + \frac{b}{\sin \theta} + \frac{c}{\cos^2 \theta} \] (3.15)
This is the case found together with its integral in [14] (see also the table of known cases in [26]).
It generalizes Kovalevskaya’s case (on the zero level of the cyclic integral) by the presence of the two singular terms in the potential.

2. For
\[ \mu = 4, \quad \alpha_0 = 1, \quad \alpha_1 = \frac{1}{8}, \quad c_0 = -\frac{3}{16}, \quad c_1 = 2, \quad c_2 = -6, \quad p = \frac{1}{4} \left( \frac{1}{\sin^2 \theta} - \cos^2 \theta \right) \] (3.16)
we obtain the potential
\[ V = \sin^2 \theta (a_1 \cos 2\varphi + a_2 \sin 2\varphi) + \frac{b}{\cos^2 \theta} + c \left( \frac{1}{\cos^4 \theta} - \frac{1}{\cos^6 \theta} \right) \]
This case is new. When \( c = 0 \) this case becomes a special version of that of Goryachev. When \( b = = c = 0 \) the remaining potential characterizes Chaplygin’s case of motion of a rigid body in a liquid. The quartic integral corresponding to (3.16) is
\[ I = \frac{\sin^8 \theta}{(1 + \sin^2 \theta)^2} \dot{\varphi}^4 + \]
\[ \frac{2 \sin^2 \theta}{(1 + \sin^2 \theta)^2} \left[ -h \sin^2 \theta + (\cos^4 \theta + \sin^2 \theta)(a_1 \cos 2\varphi + a_2 \sin 2\varphi) \right] \dot{\varphi}^2 + \]
\[ + \frac{2 \sin \theta \cos^3 \theta}{1 + \sin^2 \theta} (a_1 \sin 2\varphi - a_2 \cos 2\varphi) \dot{\varphi} - \]
\[ - (1 - \cos^4 \theta)(a_1 \sin 2\varphi - a_2 \cos 2\varphi)^2 + \]
\[ + 2(b - h)(a_1 \cos 2\varphi + a_2 \sin 2\varphi) \] (3.17)
where \( h \) in the second term of the integral should signify either the numerical value of the energy integral of the system or its expression in terms of the state variables. In the traditional Euler-Poisson variables the potential takes the form [27]
\[ V = a_1 \left( \gamma_2^2 - \gamma_1^2 \right) + 2a_2 \gamma_1 \gamma_2 + \frac{b}{\gamma_3^2} + c \left( \frac{1}{\gamma_3^4} - \frac{1}{\gamma_3^6} \right) \] (3.18)
and the integral can be written as
\[ I = \left[ p^2 - q^2 + a_1 \gamma_3^2 - \frac{b \left( \gamma_1^2 - \gamma_2^2 \right)}{\gamma_3^2} \right]^2 + \left[ pq + a_2 \gamma_3^2 - \frac{2b \gamma_1 \gamma_2}{\gamma_3^2} \right]^2 + \]
\[ + 2c \left( p^2 + q^2 \right) \left( \frac{1}{\gamma_3^4} - \frac{1}{\gamma_3^6} \right) + c \frac{\left( \gamma_1^2 + \gamma_2^2 \right)^2}{\gamma_3^{12}} (c - 2b_3^4) + \]
\[ + \frac{2c}{\gamma_3^2} \left[ a_1 \left( \gamma_1^2 - \gamma_2^2 \right) + 2a_2 \gamma_1 \gamma_2 \right] \] (3.19)
3. The choice

\[ \mu = \frac{1}{4}, \quad \alpha_0 = \alpha_1 = c_1 = c_2 = 0, \quad \alpha_2 = -\frac{1}{2}, \quad c_0 = 1, \quad p = \frac{\sqrt{2} \cos \theta}{\sqrt{\sin \theta}} \]  

(3.20)

leads to the potential

\[ V = \frac{a \cos \theta}{\sqrt{\sin \theta}} \cos \frac{\varphi}{2} + \frac{b}{\sin \theta} + \frac{c \cos \theta}{\sin^2 \theta}. \]  

(3.21)

This potential exhibits three types of singularities at \( \theta = 0, \pi \). It suffers a continuity problem on the configuration manifold of the Eulerian angles, since it has the least period of \( 4\pi \) in the angle \( \varphi \). The integral for this case is

\[ I = \frac{\sin^8 \theta}{(1 + \sin^2 \theta)^3} \frac{\dot{\varphi}^4}{4} + \frac{2 \sin^4 \theta}{(1 + \sin^2 \theta)^2} \left[ -h \sin \theta + a \sqrt{\sin \theta} \cos \theta \cos \frac{\varphi}{2} \right] \varphi^2 + \]

\[ + \frac{2a \sin^2 \theta \sin \frac{\varphi}{2}}{1 + \sin^2 \theta} \varphi \dot{\varphi} + 2ac \cos \frac{\varphi}{2} - a^2 \sin \theta \sin^2 \frac{\varphi}{2}. \]  

(3.22)

4. Finally, the choice

\[ \mu = \frac{4}{9}, \quad \alpha_0 = \alpha_1 = c_0 = c_2 = 0, \quad \alpha_2 = -\frac{1}{2}, \quad c_1 = 1, \quad p = \frac{3}{4} \left( \frac{3 \cos^4 \theta}{\sin^2 \theta} \right)^{\frac{1}{3}} \]  

leads also to the singular potential

\[ V = \frac{a \cos 2\theta \cos \frac{2}{3} \varphi}{\sin^3 2\theta} + \frac{b \cos^2 \frac{2}{3} \theta}{\sin^3 \theta} + \frac{c}{\sin^3 2\theta}. \]  

(3.23)

and the integral

\[ I = \frac{\sin^8 \theta}{(1 + \sin^2 \theta)^3} \frac{\dot{\varphi}^4}{4} + \frac{2 \sin^4 \theta}{(1 + \sin^2 \theta)^2} \left[ -h + 2a \cos^3 \theta \cos \frac{4}{3} \varphi \right] \frac{\varphi^2}{\sin \frac{4}{3} \varphi} + \frac{\sin \left( \frac{2}{3} \varphi \right)}{1 + \sin^2 \theta} \frac{\dot{\varphi}^2}{\frac{2}{3} \varphi} + \]

\[ + 4ab \cos \frac{2}{3} \varphi - \frac{a^2 \cos^3 \frac{2}{3} \theta \sin^3 \frac{2}{3} \theta}{256} \left[ 81 \cdot \frac{2^2}{3} \varphi - 1024 \cos^2 \frac{2}{3} \varphi \right]. \]  

(3.24)

Both potential and integral have period \( 3\pi \) in the variable \( \varphi \) and thus the present case cannot be used globally as an integrable model of a rigid body.

4. Conclusion

The method applied here has once more proved effective in the systematic construction of integrable mechanical systems with a polynomial second invariant, which is, in the present work, of the fourth degree in velocities. The method is augmented by some new ideas that enable systematically add new parameters in the structure of the system.

We have constructed two new many-parameter integrable systems with a quartic integral and restored several previously known results as special cases of them. Our results have the advantage
that the configuration manifold is not necessarily Euclidean. This widens the scope of applications. For example, from the new systems we have obtained, as special cases, new integrable problems of motion of a particle in different two-dimensional spaces, with positive, negative curvature and curvature of variable sign. New generalizations of the known classical integrable cases and also completely new ones with quartic integrals are found in the dynamics of a rigid body with the celebrated Kovalevskaya configuration \( A = B = 2C \).

To facilitate the solution of the system of partial differential equations, we have had to impose the condition that the force function has the structure \( U = u(\eta) + v(\eta)\Phi(\xi) \). We have considered here only the time reversible systems. Irreversible systems (systems with velocity-dependent potential) will be dealt with in a forthcoming work.

References