INTEGRABILITY OF GENERALIZED JACOBI PROBLEM

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We consider a point moving in an ellipsoid \( a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 1 \) under the influence of a force with quadratic potential \( V = \frac{1}{2}(b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2) \). We prove that the equations of motion of the point are meromorphically integrable if and only if the condition \( b_1(a_2 - a_3) + b_2(a_3 - a_1) + b_3(a_1 - a_2) = 0 \) is fulfilled.

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1. Introduction

The aim of this paper is an investigation of the following problem. A point of the unit mass moves on the surface of an ellipsoid
\[ \mathcal{E} := \{ x \in \mathbb{R}^3 \mid \langle x, Ax \rangle = 1 \}, \]
under the influence of an external force with potential
\[ V_B = \frac{1}{2} \langle x, Bx \rangle, \quad (1.1) \]
where \( A = \text{diag}(a_1, a_2, a_3), \ B = \text{diag}(b_1, b_2, b_3) \) and \( a_i, b_i \in \mathbb{R}, \ a_i > 0, \) for \( i = 1, 2, 3. \) In the above formulae \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product in \( \mathbb{R}^3. \) We call it the generalized Jacobi problem. This name is well justified—one can find exactly the same problem formulated at the end of lecture XXVIII of the famous Vorlesungen über Dynamik of C. G. J. Jacobi.

The equations of motion of the generalized Jacobi problem can be written in the following form
\[ \dot{x} = y, \quad \dot{y} = -Bx - \lambda Ax, \quad (1.2) \]
where \( \lambda \) is the Lagrange multiplier
\[ \lambda := \frac{\langle y, Ay \rangle - \langle Bx, Ax \rangle}{\langle Ax, Ax \rangle}. \quad (1.3) \]
We consider equations (1.2) on the tangent bundle
\[ T\mathcal{E} = \{ (x, y) \in \mathbb{R}^6 \mid x \in \mathcal{E}, \ \langle y, Ax \rangle = 0 \}. \]
On \( T\mathcal{E} \) equations (1.2) possess the energy integral
\[ E(x, y) = \frac{1}{2} \langle y, y \rangle + V_B(x). \]

The system has two degrees of freedom and for its integrability one additional first integral is necessary.

**Remark 1.** Let us note the following simple fact. Let \( W : \mathbb{R} \to \mathbb{R} \) be a differentiable function. The gradient of a potential \( w(x) := W(\langle x, Ax \rangle) \) is normal to the ellipsoid \( \mathcal{E}. \) Thus, for a point moving on the ellipsoid \( \mathcal{E}, \) the problems with potential \( V(x) \) and with potential \( \tilde{V}(x) = V(x) + W(\langle x, Ax \rangle) \) are exactly the same. Hence, when discussing the generalised Jacobi problem, we can assume that the cases given by matrices \( B \) and \( \tilde{B} = B + \alpha A, \ \alpha \in \mathbb{R} \) are equivalent.

As the main purpose of our investigation is to find the necessary and sufficient conditions for the integrability of the generalised Jacobi problem, let us summarize shortly all known facts.

**Geodesic motion.** When \( B = 0, \) then equations (1.2) describe a free motion of a point on an ellipsoid \( \mathcal{E}. \) The integrability of this case was discovered by Jacobi [14], [15]. He showed that the Hamilton-Jacobi equations separate in the elliptic coordinates. The additional first integral written in \( (x, y) \) variables has the following form
\[ F = \langle Ax, Ax \rangle \langle y, Ay \rangle. \]
This form of the additional first integral was found by F. Joachimsthal [16], see also [1]. It is worth noticing that \( F \) is a first integral of equations (1.2) considered as equations on \( \mathbb{R}^6(x, y), \) i.e., without the restriction to \( T\mathcal{E}. \)

**Quadratic radial potential.** A case when \( B = bE, \) where \( E \) is the unit matrix and \( b \in \mathbb{R}, \) was also investigated by Jacobi. The additional first integral for this case has the following form
\[ F = \langle Ax, Ax \rangle \{ \langle y, Ay \rangle - b \langle x, Ax \rangle \}. \]
As in the case of geodesic motion, this problem is separable in the elliptic coordinates.

**Neumann case.** For $A = E$, i.e., when the ellipsoid is a sphere, the generalized Jacobi system coincides with the system of C. Neumann [31]. As it is well known, the Neumann system is integrable. The additional first integral can be written in the following form

$$F = \langle y, By \rangle + \langle y, y \rangle \langle x, Bx \rangle + \langle Bx, Bx \rangle,$$

see [30].

**Axially symmetric case.** When the ellipsoid and the potential are axially symmetric along the same axes, then, obviously, the system is integrable. For example, if $a_1 = a_2$ and $b_1 = b_2$, then

$$F = x_1 y_2 - x_2 y_1,$$

is the additional first integral.

**Generalized Jacobi problem.** As we have already mentioned, Jacobi considered also the case of an arbitrary potential of the form (1.1). He gave the necessary and sufficient condition for separation of this potential in the elliptic coordinates. Jacobi wrote this condition in the following form

$$b_1(A_1 + \lambda_1) \left( \frac{1}{(A_1 - A_2)(A_1 - A_3)} \right) + b_2(A_2 + \lambda_1) \left( \frac{1}{(A_2 - A_1)(A_2 - A_3)} \right) + b_3(A_3 + \lambda_1) \left( \frac{1}{(A_3 - A_1)(A_3 - A_2)} \right) = 0,$$

where $A_i = a_i^{-1}$ and $\lambda_1$ denotes the first elliptic coordinate. He stopped his discussion exactly at this point. Let us notice that the ellipsoid $E$ is given by $\lambda_1 = 0$, so the Jacobi condition can be rewritten in the following form

$$W_J := \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_1(b_2 - b_3) + a_2(b_3 - b_1) + a_3(b_1 - b_2) = 0.$$

If ellipsoid $E$ is not a sphere, then the above condition is equivalent to the following

$$b_i = b + a a_i, \quad \text{for } i = 1, 2, 3,$$

or

$$B = b E + a A.$$

Thus, by Remark 1, the generalized Jacobi case is separable in the elliptic coordinates if and only if the potential is equivalent to the quadratic radial potential.

It is important to notice that for all the above integrable cases condition (1) is satisfied. The main result of this paper is following.

**Theorem 1.** The generalized Jacobi problem is integrable with a meromorphic first integral if and only if

$$W_J = a_1(b_2 - b_3) + a_2(b_3 - b_1) + a_3(b_1 - b_2) = 0.$$
2. Outline of the Morales-Ramis theory

Below we describe only the basic notions and facts concerning the Morales-Ramis theory. We also mention some basic facts of the Ziglin theory because the Morales-Ramis theory is its natural extension. Both these theories take their origins in works of S.W. Kovalevskaya and A.M. Lyapunov and their main idea lies in a deep connection between the properties of solutions of the analyzed system in the complex time plane, and the existence of first integrals of this system. Interested reader can find a more detailed and formal presentation of the Ziglin and Morales-Ramis theories e.g. in [4], [27], [28], [42], [43], [2].

Let us consider a system of holomorphic differential equations
\[
\frac{dx}{dt} = v(x), \quad t \in \mathbb{C}, \quad x \in M, \tag{2.1}
\]
on a complex manifold $M$. We assume that a non-constant particular solution $\varphi(t)$ of system (2.1) is known. Its maximal analytic continuation defines a Riemann surface $\Gamma$ with the local coordinate $t$.

The variational equations (VEs) along $\varphi(t)$ have the form
\[
\dot{x} = T(v)\xi, \quad T(v) = \frac{\partial v(\varphi(t))}{\partial x}, \quad \xi \in T_{\Gamma}M. \quad (2.2)
\]

If system (2.1) is Hamiltonian with $m$ degrees of freedom, i.e. $n = 2m$, then the order of VEs can be reduced by two. First we use the fact that a Hamiltonian system has at least one first integral, namely Hamiltonian $H$, thus we can restrict system (2.1) to the manifold $M_\varepsilon = \{x \in M \mid H(x) = \varepsilon\}$, where $\varepsilon = H(\varphi(t))$. Then we consider the induced system on the normal bundle $N := T_{\Gamma}M_\varepsilon/TT$ of $\Gamma$
\[
\dot{\eta} = \pi_*(T(v)(\pi^{-1}\xi)), \quad \eta \in N. \quad (2.2)
\]

Here $\pi : T_{\Gamma}M_\varepsilon \rightarrow N$ is the projection. The system of $n - 2$ equations obtained in this way is called the normal variational equations (NVEs).

With the linear system (2.2) we can associate the monodromy group $\mathcal{M}$. The monodromy group of (2.2) is the image of the fundamental group obtained in the process of analytic continuation of a local solution of (2.2) along loops with an arbitrary base point. For Hamiltonian systems it is a subgroup of $\text{Sp}(2(m-1), \mathbb{C})$. A non-constant rational function $f(z)$ of variables $z = (z_1, \ldots, z_{2(m-1)})$ is called an integral of the monodromy group, if $f(Mz) = f(z)$ for all $M$ belonging to $\mathcal{M}$.

If the dynamical system (2.1) has $k$ meromorphic functionally independent first integrals, then also system (2.2) of NVEs has $k$ meromorphic functionally independent first integrals. Next, the invariance of the $k$ first integrals of variational equations with respect to the continuations of their solutions implies the existence of $k$ rational first integrals of $\mathcal{M}$ [42], [43]. This basic result of the Ziglin theory is used to formulate a necessary condition for the integrability [42].

The Ziglin works found a lot of continuations and many important applications [3], [8], [9], [11], [12], [13], [34], [35], [39], [40], [41]. But during the applications of the Ziglin theory two important technical problems appear

- calculation of the monodromy group. Since a monodromy group is not a Lie group, infinitesimal methods cannot be applied. As a result, this group is known only for a few differential equations e.g. the hypergeometric equation or the Jordan-Pochhammer equation.

- finding a non-resonant element of the monodromy group. This is some technical assumption which appears in the Ziglin theory. For systems with more than two degrees of freedom it is not easy to check if it is satisfied.

For these reasons the Ziglin theory works mainly for systems with two degrees of freedom.
With the variational equations is related another group, called a differential Galois group which is a Lie group. The differential Galois group \( \mathcal{G} \) of (2.2) is a matrix group acting on the fundamental solutions of (2.2) which does not change polynomial relations among them, for a precise definition see e.g. [17], [33], [37]. It is an algebraic group, and for Hamiltonian systems it is a subgroup of \( \text{Sp}(2(m-1), \mathbb{C}) \). Thus \( \mathcal{G} \) is a union of disjoint connected components. One of them containing the identity is called the identity component of \( \mathcal{G} \) and denoted by \( \mathcal{G}^0 \). Between \( \mathcal{M} \) and \( \mathcal{G} \) there exists a relation \( \mathcal{M} \subset \mathcal{G} \).

The integrals of the differential Galois group are similarly defined as for the monodromy group. If system (2.1) has \( k \) meromorphic functionally independent first integrals, then the differential Galois group of NVEs has \( k \) functionally independent rational first integrals [27].

A replacement of the monodromy group by the differential Galois group yields a more powerful tool for the integrability analysis because

- there are differential equations with a trivial monodromy group and with a non-trivial differential Galois group,
- the differential Galois group is an algebraic group and, in particular, it is a Lie group, thus all algebraic calculations are easier. In particular, for a second order differential equation with rational coefficients the Kovacic algorithm [18] determines completely its differential Galois group,
- there is no resonance assumptions.

The existence of the Lie algebra of the differential Galois group enables us to formulate for Hamiltonian systems the necessary conditions for integrability in the Liouville sense in a very convenient way, by means of the properties of the identity component of the differential Galois group of NVEs.

**Theorem 2 (Morales-Ruiz and Ramis [27]).** Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of a phase curve \( \Gamma \). Then the identity component of the differential Galois group of the normal variational equations associated with \( \Gamma \) is Abelian.

The above theorem yields an effective tool for proving the non-integrability, see e.g. [7], [26], [19], [20], [22], [21], [24], [25], [23], [27], [29], [36].

**3. Particular solutions and normal variational equations**

Without loss of generality, we can assume that \( a_1 \geq a_2 \geq a_3 \geq 0 \). Moreover, Remark 1 allows us to assume that \( b_1 \neq 0 \). From now on we work with a complexified system (1.2), i.e. we assume that \( (x, y) \in \mathbb{C}^6 \) and \( t \in \mathbb{C} \).

It is easy to observe that the following three manifolds

\[
\Pi_k = \{(x, y) \in \mathbb{C}^6 | x_k = 0, \quad y_k = 0\}, \quad k = 1, 2, 3,
\]

are invariant with respect to the flow generated by (1.2). System (1.2) restricted to \( \Xi_k := \Pi_k \cap T\mathcal{E} \) yields a system with one degree of freedom which can be integrated explicitly. Thus we obtain three families of particular solutions of (1.2) which will be used in our non-integrability proofs.

Let us consider the constant energy level

\[
\Sigma_h := \{(x, y) \in T\mathcal{E} | E(x, y) = h\}.
\]

We assume that \( h \in \mathbb{C} \) is such that for \( k = 1, 2, 3 \) intersection \( \Sigma_k \cap \Xi_k \) is non-empty and contains a solution

\[
x = \varphi(t), \quad y = \varphi(t), \quad t \in U \subset \mathbb{C},
\]
different from an equilibrium. A maximal analytical continuation of solution (3.1) lying in \( \Xi_k \) yields the Riemann surface which we denote by \( \Gamma_k^h \). Linear variational equations along this solution can be written in the following form

\[
\ddot{\xi} + [B + \lambda A] \xi + \left( \langle \nabla_x \lambda, \xi \rangle + \langle \nabla_y \lambda, \dot{\xi} \rangle \right) A \varphi(t) = 0,
\]

where \( \lambda, \nabla_x \lambda \) and \( \nabla_y \lambda \) are calculated at \( (x, y) = (\varphi(t), \dot{\varphi}(t)) \), and \( \xi = (\xi_1, \xi_2, \xi_3) \). Since the considered solution lies on \( \Xi_k \), we have \( \varphi_k(t) \equiv 0 \) and \( \dot{\varphi}_k(t) \equiv 0 \). Thus, the normal variational equation for this solution corresponds to the \( k \)-th component of \( \xi \) and can be written in the following form

\[
\ddot{\xi}_k + [b_k + \lambda a_k] \xi_k = 0. \tag{3.2}
\]

To simplify the further exposition we restrict our attention to the case \( k = 3 \). Then note that the Riemann surface \( \Gamma_3^h \) is an algebraic curve determined by the following system

\[
\begin{align*}
a_1 x_1^2 + a_2 x_2^2 &= 1, \quad x_3 = 0, \quad y_3 = 0, \\
a_1 x_1 y_1 + a_2 x_2 y_2 &= 0, \quad y_1^2 + y_2^2 + b_1 x_1^2 + b_2 x_2^2 &= 2h.
\end{align*} \tag{3.3}
\]

Thus our NVE (3.2) can be considered as a differential equation on the algebraic curve determined by this system. Since there are many results concerning solvability of differential equations on Riemann sphere \( \mathbb{C}P^1 \), we transform equation (3.2) into a second order differential equation with rational coefficients by means of a finite covering transformation. We can do this e. g. by means of the following change of variable

\[
t \longrightarrow z = \varphi_1(t)^2.
\]

We see that this transformation really rationalizes equation (3.2) because after this change of variable \( \lambda = \lambda(\varphi(t), \dot{\varphi}(t)) \) is a rational function of \( z \) and we denote it \( \lambda(z) \). Moreover, we have

\[
\begin{align*}
\frac{d}{dt} &= 2\varphi_1(t)\dot{\varphi}_1(t) \frac{d}{dz}, \\
\frac{d^2}{dt^2} &= \left[2\ddot{\varphi}_1(t)^2 - 2(\lambda(z)a_1 + b_1)z\right] \frac{d}{dz} + 4z\ddot{\varphi}_1(t) \frac{d^2}{dz^2}.
\end{align*} \tag{3.4}
\]

To obtain the last relation we used the explicit form of the right hand side of equation (1.2) for \( \ddot{x}_1 \). From (3.3) it follows also that \( \varphi_1(t)^2 = y_1^2 \) is a rational function of \( z \), which we denote as \( y_1(z)^2 \). Using (3.4) we can rewrite normal variational equation (3.2) with the new variable in the following form

\[
\frac{d^2}{dz^2} + p(z) \frac{d}{dz} + q(z) \xi_3 = 0, \tag{3.5}
\]

where

\[
p(z) = \frac{1}{2} \left( \frac{1}{2} - \frac{\lambda(z)a_1 + b_1}{y_1(z)^2} \right), \quad q(z) = \frac{\lambda(z)a_3 + b_3}{4zy_1(z)^2}.
\]

Making the following change of the dependent variable

\[
w = \xi_3 \exp \left[ \frac{1}{2} \int_{z_0}^{z} p(z) \, dz \right],
\]

we transform (3.5) into the standard reduced form

\[
\frac{d^2 w}{dz^2} = \bar{\tau}(z) w, \tag{3.6}
\]
where
\[ \tilde{r}(z) = -q(z) + \frac{1}{4} p(z)^2 + \frac{1}{2} \frac{dp(z)}{dz}. \]

Because \( a_1 \), as the reciprocal of the semi-axis, is different from zero, we can perform rescaling \( z \to z/a_1 \), and we obtain an equation of the form (3.6) with the rational coefficient
\[
\frac{d^2 w}{dz^2} = r(z) w, \quad r(z) = \frac{1}{a_1^2} \tilde{r}(z/a_1). \tag{3.7}
\]

This rescaling enables us to reduce the number of parameters and now rational function \( r(z) \) depends only on five parameters, namely
\[
\alpha_k = \frac{a_k}{a_1}, \quad \beta_k = \frac{b_k}{b_1}, \quad k = 2, 3, \quad \text{and} \quad \mu = 2h \frac{a_1}{b_1}.
\]

As from now on parameter \( \mu \) is related with the chosen energy level, we write \( \Gamma_i^\mu \) instead of \( \Gamma_i^h \). Let us define also
\[
W = W(\alpha_2, \alpha_3, \beta_2, \beta_3) := \frac{W_j}{a_1 b_1} = (\beta_2 - \beta_3) + \alpha_2(\beta_3 - 1) + \alpha_3(1 - \beta_2),
\]
and
\[ \mathcal{P} := \{(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathbb{R}^4 \mid 1 \geq \alpha_2 \geq \alpha_3, \quad W(\alpha_2, \alpha_3, \beta_2, \beta_3) \neq 0 \}. \]

Now Theorem 1 can be reformulated in the following way.

**Theorem 3.** For \( (\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P} \), the generalized Jacobi problem does not admit an additional meromorphic first integral.

Let us make here several remarks. First of all, in order to prove our theorem we have to investigate the differential Galois group of three NVEs calculated for three phase curves which lie on the same energy level. The number of singular points and their nature depends on the parameters of the problem, as well as on the chosen energy value. This fact complicates technically our proof. Namely, we should investigate several cases separately. Generally, we have to distinguish two cases of the non-symmetric and symmetric ellipsoid. Then, in both cases, we consider the generic case and several non-generic cases. According to this scheme, we represent the parameters’ space \( \mathcal{P} \) of our problem as a disjoint union of two sets:

\[ \mathcal{P} = \mathcal{P}_T \cup \mathcal{P}_S, \quad \mathcal{P}_S = \mathcal{P} \setminus \mathcal{P}_T, \]

where
\[ \mathcal{P}_T := \{(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P} \mid \alpha_2 \neq 1, \quad \alpha_3 \neq 1, \quad \alpha_2 \neq \alpha_3 \}. \]

Then, in each part of \( \mathcal{P} \) we distinguish its generic and non-generic components. Thus, we represent \( \mathcal{P}_T \) as
\[ \mathcal{P}_T = \mathcal{P}_{TG} \cup \mathcal{P}_{TN}, \quad \mathcal{P}_{TN} = \mathcal{P}_T \setminus \mathcal{P}_{TG}, \]

where
\[ \mathcal{P}_{TG} := \{(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_T \mid \alpha_2 \neq \beta_2, \quad \alpha_3 \neq \beta_3, \quad \alpha_2 \beta_3 \neq \alpha_3 \beta_2 \}. \]

Similarly, we decompose \( \mathcal{P}_S \) as
\[ \mathcal{P}_S = \mathcal{P}_{SG} \cup \mathcal{P}_{SN}, \quad \mathcal{P}_{SN} = \mathcal{P}_S \setminus \mathcal{P}_{SG}, \]

where
\[ \mathcal{P}_{SG} := \{(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_S \mid \alpha_2 \neq \beta_2, \quad \alpha_3 \neq \beta_3, \quad \alpha_2 \beta_3 \neq \alpha_3 \beta_2 \}. \]

In the next four sections we present separate proofs of Theorem 3 for components \( \mathcal{P}_{TG}, \mathcal{P}_{TN}, \mathcal{P}_{SG} \) and \( \mathcal{P}_{SN} \), respectively.
4. Tri-axial generic case

In this section we assume that \((\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{TG}\). Under this assumption NVE (3.7) along phase curve \(\Gamma^\mu_3\) is Fuchsian with five regular singular points over \(\mathbb{CP}^1\), namely

\[
z_1 = 0, \quad z_2 = 1, \quad z_3 = \frac{\beta_2 - \mu \alpha_2}{\beta_2 - \alpha_2}, \quad z_4 = \frac{\alpha_2}{\alpha_2 - 1}, \quad z_5 = \infty.
\]

Note that the position of \(z_3\) depends on the chosen energy value \(\mu\). Thus, to avoid possible confluences of \(z_3\) with other points, we assume that

\[
\mu \notin \left\{ \frac{1}{\alpha_2}, \frac{1}{\beta_2}, \frac{\alpha_2 - \beta_2}{\alpha_2 - \alpha_3}, \frac{\alpha_3 - \beta_3}{\alpha_3(\alpha_3 - 1)}, \frac{\alpha_2^2 \beta_3 - \alpha_3^2 \beta_2}{\alpha_2 \alpha_3(\alpha_2 - \alpha_3)} \right\}.
\]

The differences of exponents \(\Delta_i\) at singular points \(z_i\) (for definition, see Appendix A) are equal to

\[
\Delta_i = \frac{1}{2}, \quad \text{for} \quad i = 1, 2, 3, \quad \Delta_4 = \frac{3}{2},
\]

\[
\Delta_5 = \delta_3 := \sqrt{\frac{\alpha_3(1 - \beta_2) + \beta_3(\alpha_2 - 1)}{\alpha_2 - \beta_2}}.
\]

The calculations of NVE along the phase curve \(\Gamma^\mu_3\) proceed along the same line. We start from (3.2) with \(k = 1\). The change of variable \(t \to z = \varphi_2(t)^2\) transforms this equation defined on the algebraic curve into a Fuchsian equation of the form (3.7) on Riemann sphere \(\mathbb{CP}^1\) with five regular singular points

\[
z_1 = 0, \quad z_2 = \alpha_2^{-1}, \quad z_3 = \frac{\beta_3 - \mu \alpha_3}{\alpha_2 \beta_3 - \alpha_3 \beta_2}, \quad z_4 = \frac{\alpha_3}{\alpha_2(\alpha_3 - \alpha_2)}, \quad z_5 = \infty.
\]

The differences of exponents at these singular points are the same as for NVE related to \(\Gamma^\mu_3\) except for \(z_5 = \infty\), for which we have

\[
\Delta_5 = \delta_1 := \sqrt{\frac{\alpha_2 - \beta_2 + \beta_3 - \alpha_3}{\beta_3 \alpha_2 - \beta_2 \alpha_3}}.
\]

In the similar way we obtain NVE along \(\Gamma^\mu_4\). Again, after the transformation \(t \to z = \varphi_3(t)^2\) we obtain a Fuchsian equation with five singular points over \(\mathbb{CP}^1\)

\[
z_1 = 0, \quad z_2 = \alpha_3^{-1}, \quad z_3 = \frac{1 - \mu}{\alpha_3 - \beta_3}, \quad z_4 = \frac{1}{\alpha_3(1 - \alpha_3)}, \quad z_5 = \infty.
\]

The differences of exponents are the same as for the previous NVEs except for \(z_5 = \infty\), for which we have

\[
\Delta_5 = \delta_2 := \sqrt{\frac{\alpha_3(3 - \beta_3 - 1) + \beta_2(1 - \alpha_3)}{\beta_3 - \alpha_3}}.
\]

Now we prove the following lemma.

**Lemma 1.** If there exists \(k \in \{1, 2, 3\}\) such that

\[
\delta_k \neq \frac{p}{q}, \quad p \in \mathbb{N}, \quad q = 1, \ldots, 6,
\]

then the generalized Jacobi problem does not admit an additional meromorphic first integral.

**Proof.** Assume that the problem is integrable. Then, by Theorem 2, for each particular solution, the identity component \(\mathcal{G}^0\) of the differential Galois group of the respective NVE is Abelian. We show
that if condition (4.4) is satisfied, then, for NVE corresponding to $\Gamma_k^\mu$, we have \( \mathcal{G} = \mathcal{G}^0 = \text{SL}(2, \mathbb{C}) \), except for some specific value of $\mu$ which can be excluded.

We assume that $k$ in the statement of the lemma is 3, so we consider NVE corresponding to $\Gamma_3^\mu$. From the proof given below, it is clear that the same arguments apply when $k = 1$ or $k = 2$. Our proof is based on a direct application of the Kovacic algorithm, see Appendix A and [18].

As NVE is Fuchsian, all necessary conditions formulated in Lemma 13 are satisfied, and all four possibilities given by Lemma 12 can occur. Thus, we have to follow the Kovacic algorithm from Case I to Case III.

**Case I.** We check if $\mathcal{G}$ conjugates to a triangular subgroup of $\text{SL}(2, \mathbb{C})$, or, equivalently, if (3.7) has a solution of the form $w = P \exp \int \omega$, where $P \in \mathbb{C}[z]$ and $\omega \in \mathbb{C}(z)$. According to the Kovacic algorithm for each singular point we calculate the auxiliary sets $E_i$ using formulae (9.4) and (9.5) from Appendix A. We obtain

\[
E_5 = \left\{ \frac{1}{2}(1 \pm \delta_3) \right\}, \quad E_4 = \left\{ -\frac{1}{4}, \frac{5}{4} \right\},
\]

\[
E_i = \left\{ \frac{1}{4}, \frac{3}{4} \right\}, \quad \text{for } i = 1, 2, 3.
\]

Let $E = \prod E_i$ be the Cartesian product of all $E_i$. Then we determine the degree of polynomial $P$ for some $e = (e_1, \ldots, e_5) \in E$ in the following way

\[
\deg P := d(e) := e_5 - \sum_{i=1}^{4} e_i.
\]

However, for an arbitrary $e \in E$ we have

\[
d(e) = -\frac{l}{k} \pm \frac{1}{2} \delta_3, \quad k = 1, 2, 4, \quad l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.
\]

Thus, if $\delta_3$ satisfies (4.4), then $d(e) \notin \mathbb{N}_0$, and Case I cannot occur.

**Case II.** We check if $\mathcal{G}$ can be an imprimitive subgroup of $\text{SL}(2, \mathbb{C})$, or, equivalently, if there exists a solution of the form $w = \exp \int \omega$ where $\omega$ belongs to the quadratic extension of $\mathbb{C}(x)$. Now, by formulae (9.6) and (9.7) sets $E_i$ are the following

\[
E_1 = E_2 = E_3 = \{1, 2, 3\}, \quad E_4 = \{-1, 2, 5\},
\]

and

\[
E_5 = \{2, 2 \pm 2\delta_3 \} \cap \mathbb{Z} = \{2\},
\]

because $\delta_3$ satisfies (4.4).

Now, we look for $e \in E$ such that $d_2(e) := d(e)/2 \notin \mathbb{N}_0$. We have only one such element, namely $e = (1, 1, 1 - 1, 2)$ yielding $d_2(e) = 0$. Thus, we have to check if there exists a monic polynomial $P$ of degree 0 satisfying (9.9). Using a computer algebra system we can check that such solution exists only if

\[
\mu = \frac{\beta_2 - \alpha_2^2}{\alpha_2(1 - \alpha_2)}.
\]

(4.5)

Thus, for all values of $\mu$ different from this one, Case II cannot occur.

**Case III.** We check if $\mathcal{G}$ can be a finite subgroup of $\text{SL}(2, \mathbb{C})$, or, equivalently, if there exists a solution of the form $w = \exp \int \omega$ where $\omega$ belongs to an algebraic extension of $\mathbb{C}(z)$ of degree $n = 4, 6$ or 12. Each value of $n$ must be considered separately.

**Sub-case** $n = 4$. Now by formulae (9.10) and (9.11) the sets $E_i$ are the following

\[
E_1 = E_2 = E_3 = \{3, 6, 9\}, \quad E_4 = \{-3, 6, 15\}, \quad E_5 = \{6\}.
\]
To calculate $E_i$ we used condition (4.4). It is easy to check that there is no $e \in E$ such that $d_4(e) := d(e)/3 \in \mathbb{N}_0$. Thus this sub-case cannot occur.

**Sub-case** $n = 6$. Now we calculate

$$E_1 = E_2 = E_3 = \{3, 4, 5, 6, 7, 8, 9\}, \quad E_4 = \{-3, 0, 3, 6, 9, 12, 15\}, \quad E_5 = \{6\},$$

and there is only one $e \in E$ such that $d_6(e) := d(e)/2 \in \mathbb{N}_0$, namely $e = (3, 3, 3, -3, 6)$, which yields $d_6(e) = 0$. In the next step we have to check if the polynomial $P = 1$ satisfies the equation defined in a recursive way in (9.12), if it does not satisfy it, then this sub-case is excluded. Using computer algebra we checked that $P = 1$ fulfills this equation only for one value of $\mu$ given by (4.5). Thus for all values of $\mu$ except for this one the sub-case $n = 6$ cannot occur.

**Sub-case** $n = 12$. Sets $E_i$ are the same as in the previous sub-case. Again there is only one $e \in E$, exactly the same as in the previous sub-case, such that $d_{12}(e) := d(e) \in \mathbb{N}_0$. With the help of a computer algebra program, we checked that an appropriate equation has no polynomial solution of degree 0 except for the value of $\mu$ given by (4.5). Thus, this sub-case, as well as the whole Case III, must be excluded.

From the above considerations it follows that for generic values of $\mu$ the considered NVE has no Liouvillian solution and its differential Galois group is $SL(2, \mathbb{C})$. Thus, for the generic values of $\mu$, the identity component of the differential Galois group of NVE is not Abelian. This contradicts our assumption that the system is integrable.

**Remark 2.** It should be noticed that the statement of the above lemma is valid for all values of $(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}$ for which

1. at least one NVE is Fuchsian with five singularities, and
2. the finite singularities have the differences of exponents $(1/2, 1/2, 1/2, 3/2)$.

The main result of this section is the following lemma which shows that Theorem 3 is valid when we restrict the parameters to $\mathcal{P}_{TG}$.

**Lemma 2.** Assume that $(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{TG}$. Then the generalized Jacobi problem does not admit an additional meromorphic first integral.

**Proof.** Assume that for some parameters’ values from $\mathcal{P}_{TG}$ the system is integrable. Then, from Lemma 1, it follows that in cases suspected of the integrability

$$\delta_k^2 \in J := \left\{ r \in \mathbb{Q} \mid r = p^2/q^2, \quad p \in \mathbb{N}, \quad q \in \{1, 2, 3, 4, 5, 6\} \right\}, \quad (4.6)$$

for $k = 1, 2, 3$. Thus $\delta_k^2 \geq 0$, for $k = 1, 2, 3$. From the definition of $\delta_k^2$, see (4.1), (4.2) and (4.3), it follows that

$$\delta_k^2 \delta_2^2 \delta_3^2 - (\delta_1^2 + \delta_2^2 + \delta_3^2) + 2 = 0. \quad (4.7)$$

Notice that if $\delta_3^2 = 1$ for some $i \in \{1, 2, 3\}$, then $W = 0$. Thus, $\delta_i^2 \neq 1$, for $i = 1, 2, 3$. Furthermore, from relation (4.7) it follows that always one of $\delta_i^2$, e.g. $\delta_1^2$ is smaller than 1. Together with condition $\delta_1^2 \in J$ this implies

$$\delta_1^2 \in J_0 = \left\{ 0, 1, 1, 1, 4, 4, 1, 9, 9, 1, 1, 4, 4, 9, 9, 1, 16, 16, \frac{25}{25}, \frac{25}{25}, \frac{25}{25}, \frac{36}{36} \right\}. \quad \text{J_0}$$(4.8)

For fixed $\delta_1^2 \in J_0 \setminus \{0\}$, relation (4.7) is an equation of hyperbola $\mathcal{H}$. Its vertical and horizontal asymptotes are given by $\delta_2^2 = \delta_1^{-2}$ and $\delta_3^2 = \delta_1^{-2}$, respectively, see Figure 1. We see that for each $\delta_1^2 \in J_0$ its asymptotic value $\delta_1^{-2}$ is in $J$. But in this case relation (4.7) yields condition $-\delta_1^{-2}(\delta_1^{-2} - 1)^2 = 0$ which cannot be satisfied for $\delta_1^{-2} \neq 1$.

Let us define

$$\delta_{2,a}^2 = \delta_{3,a}^2 = \min_{\delta \in J} \{\delta^2 \mid \delta^2 - \delta_1^{-2} > 0\}, \quad (4.9)$$

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Fig. 1. Hyperbola $H$ defined by (4.7) for a fixed value of $\delta_1^2$. Dashed lines denote the asymptotes. See the text for explanations.

and let us take such $\delta_2^2_{\text{max}}$ and $\delta_3^2_{\text{max}}$ that points $(\delta_2^2_{\text{a}}, \delta_3^2_{\text{max}})$ and $(\delta_2^2_{\text{max}}, \delta_3^2_{\text{a}})$ lie on hyperbole $H$, see Figure 1. Now, it is obvious that if $(\delta_2^2, \delta_3^2) \in J \times J$ satisfy relation (4.7), then $0 < \delta_2^2 \leq \delta_2^2_{\text{max}}$ and $0 < \delta_3^2 \leq \delta_3^2_{\text{max}}$. This means that for each $\delta_1^2 \in J_0$ there is only a finite set of pairs $(\delta_2^2, \delta_3^2) \in J \times J$ which can satisfy relation (4.7). For each $\delta_1^2 \in J_0$ and the corresponding finite sets of pairs $(\delta_2^2, \delta_3^2) \in J \times J$ we checked all possibilities. It appears that $(\delta_1^2, \delta_2^2, \delta_3^2) \in J_0 \times J \times J$ satisfies relation (4.7) only at $(1/25, 49, 49)$.

For $\delta_1^2 = 0$ relation (4.7) determines a straight line $\delta_3^2 = 2 - \delta_2^2$, thus we have only a few values $\delta_2^2 \in J$ for which $\delta_3^2 \geq 0$, but none lies on this line.

Symmetry of relation (4.7) implies that in cases $0 \leq \delta_2^2 < 1$ and $0 \leq \delta_3^2 < 1$ relation (4.7) is satisfied only at $(49,1/25,49)$ and $(49,49,1/25)$ respectively. Thus, there are only three cases when the condition of Lemma 1 is not fulfilled. To complete the proof we consider these cases in more detail.

First we note that the values of $(\delta_1^2, \delta_2^2, \delta_3^2)$, corresponding to the above cases, yield relations between parameters $\alpha_2, \alpha_3, \beta_2, \beta_3$:

- $(1/25, 49, 49)$ yields $\alpha_2 = 50 - \alpha_3, \quad \beta_2 = 50 - \beta_3$,
- $(49,1/25,49)$ yields $\alpha_3 = 50\alpha_2 - 1, \quad \beta_3 = 50\beta_2 - 1$,
- $(49,49,1/25)$ yields $\alpha_2 = 50\alpha_3 - 1, \quad \beta_2 = 50\beta_3 - 1$.

Taking into account the symmetry of the above relations, we can study only one case. We put $(\delta_1^2, \delta_2^2, \delta_3^2) = (49, 49, 1/25)$ and apply the Kovacic algorithm. To this end we consider NVE corresponding to $\Gamma_3^3$ and use calculations performed in the proof of Lemma 1.

Case I. We have the following formula for the degree of polynomial $P$

$$d(\varepsilon) = -\frac{l}{k} \pm \frac{1}{10^l}, \quad k = 1, 2, 4, \quad l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$  

Thus, Case I cannot occur.
Case II. In this case the sets $E_i$ ($i = 1, \ldots, 5$) are identical to the sets of the corresponding Case II considered in the proof of Lemma 1. It follows that Case II cannot occur.

Case III. Sub-case $n = 4$ and Sub-case $n = 6$. We have the same situation as in Case II. Thus, either of the sub-cases cannot occur.

Sub-case $n = 12$. In this case a special study is necessary. First we calculate

$$E_1 = E_2 = E_3 = \{3, 4, 5, 6, 7, 8, 9\}, \ E_4 = \{-3, 0, 3, 6, 9, 12, 15\}, \ E_5 = \{5, 6, 7\}.$$ 

There are the following possibilities

- The elements $(3, 3, 3, -3, 6), (4, 3, 3, -3, 7), (3, 4, 3, -3, 7), (3, 3, 4, -3, 7)$ yield $d_{12}(\epsilon) := d(\epsilon) = 0$. Our calculations have shown that with the exception of the value of $\mu$ given by (4.5) there is no polynomial $P = 1$ satisfying (9.12).

- The element $(3, 3, 3, -3, 7)$ yields $d_{12}(\epsilon) := d(\epsilon) = 1$. With the help of a computer algebra program we checked that a polynomial of degree 1 satisfying (9.12) exists only for the value of $\mu$ given by (4.5).

Thus, under our assumptions, this sub-case, as well as the whole Case III, must be excluded. This completes our proof.

\section{5. Non-generic three-axial case}

In this section we assume that $(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{\text{TN}}$. Thus we assume that the ellipsoid is three-axial and

$$\alpha_2 = \beta_2 \quad \text{or} \quad \alpha_3 = \beta_3 \quad \text{or} \quad \alpha_2\beta_3 = \alpha_3\beta_2.$$ 

It is easy to observe that if two of the above equalities are satisfied, then $W = 0$. Thus, we have three sub-cases when only one of the above equalities is satisfied. Their analysis is similar, so we present here only the analysis of the sub-case $\beta_2 = \alpha_2$.

Having in mind the above assumptions, we consider NVE (3.7) along $\Gamma_{q_1}'$. Now it possesses three finite singularities $z_1 = 0$, $z_2 = 1$, $z_3 = \alpha_2/(\alpha_2 - 1)$, and the differences of exponents at these points are

$$\Delta_1 = \Delta_2 = 1/2, \quad \Delta_3 = 3/2,$$

respectively. According to our parameters’ restrictions, we have $\alpha_2 \neq 1$ and $\beta_3 \neq \alpha_3$, and these restrictions guarantee that the order of infinity is equal to 3. To check if the identity component of $\mathcal{G}$ for this NVE is Abelian, we also apply the Kovacic algorithm. Our NVE is not Fuchsian and an analysis of the necessary conditions formulated in Lemma 13 shows that only Cases II and IV of Lemma 12 are possible. We check if Case II of the Kovacic algorithm can occur. According to (9.6) and (9.7) the auxiliary sets $E_i$ are following

$$E_1 = E_2 = \{1, 2, 3\}, \quad E_3 = \{-1, 2, 5\}, \quad E_4 = \{1\}.$$ 

Thus we have only one element $\epsilon = (1, 1, -1, 1) \in \prod_{i=1}^4 E_i$ yielding $d(\epsilon) = 0 \in \mathbb{N}_0$. The corresponding rational function $\theta$ (see (9.8)) has the form

$$\theta(z) = \frac{1}{2} \left( \frac{1}{z - z_1} + \frac{1}{z - z_2} - \frac{1}{z - z_3} \right),$$

and equation (9.9) determining the existence of polynomial $P = 1$ yields the condition $(\alpha_2 - 1)\alpha_3 = 0$, which for $\alpha_3 \neq 0$ and $\alpha_2 \neq 1$ is not satisfied. Thus the differential Galois group of the analyzed NVE is $\text{SL}(2, \mathbb{C})$ and, in particular, is not Abelian.

Making similar calculations for the two remaining sub-cases $\alpha_3 = \beta_3$ and $\alpha_2\beta_3 = \alpha_3\beta_2$, we obtain the same conclusions: the differential Galois group of the respective NVE is $\text{SL}(2, \mathbb{C})$ and, in particular, is not Abelian. Thus the system is not integrable for the specified domain of the parameters’ space. In this way we proved the following.
Lemma 3. Assume that \((\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{TN}\). Then the generalized Jacobi problem does not admit an additional meromorphic first integral.

6. Symmetric generic case

In this section we assume that \((\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{SG}\). Thus, the ellipsoid is axially symmetric (but not spherically symmetric), and we have to investigate separately two sub-cases:

\[ \alpha_2 = 1 \quad \text{and} \quad \alpha_3 \neq 1; \quad \alpha_2 = \alpha_3 \neq 1. \]

The proofs of non-integrability of these two described sub-cases are different. We present them in separated sub-sections.

6.1. Sub-case \(\alpha_2 = 1\)

When \(\alpha_2 = 1\) and \(\alpha_3 \neq 1\), our assumption that \((\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{SG}\) implies that

\[ \beta_2 \neq 1 \quad \text{and} \quad \alpha_3 \neq \beta_3 \quad \text{and} \quad \beta_3 \neq \alpha_3 \beta_2. \]

First, we prove the following lemma.

Lemma 4. Assume that \((\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{SG}\) and \(\alpha_2 = 1\). If the generalized Jacobi problem admits an additional meromorphic first integral, then \(\alpha_3 = 3/8\).

Proof. Under the assumptions of this lemma, NVE along phase curve \(\Gamma_3^\mu\), for \(\mu \notin \{1, \beta_2\}\), has four regular singular points:

\[ z_1 = 0, \quad z_2 = 1, \quad z_3 = \frac{\beta_2 - \mu}{\beta_2 - 1}, \quad z_4 = \infty. \]

The differences of exponents at these points are the following

\[ \Delta_1 = \Delta_2 = \Delta_3 = \frac{1}{2}, \quad \Delta_4 := \delta_3 = \frac{1}{2} \sqrt{1 + 8 \alpha_3}, \]

thus this NVE is a Lamé equation. Indeed, if we take the non-reduced form (3.5) of the analyzed NVE and apply a linear change of variable \(z = gv + p\), where \(g \in \mathbb{C}^*\) is an arbitrary parameter and

\[ p = \frac{\mu + 1 - 2 \beta_2}{3(1 - \beta_2)}, \]

then NVE takes an algebraic form of the Lamé equation, see Appendix B,

\[ \frac{d^2 \xi}{dv^2} + \frac{f'(v)}{2f(v)} \frac{d \xi}{dv} - \frac{Av + B}{f(v)} \xi = 0, \quad (6.1) \]

with parameters

\[ A = 2\alpha_3, \quad B = \frac{3 \beta_3 + \alpha_3 \mu - 2(\beta_2 + 1)}{3g(\beta_2 - 1)}. \]

Here polynomial \(f(v)\) has the form

\[ f(v) = 4v^3 - g_2v - g_3, \]

where

\[ g_2 = \frac{4[\beta_2^3 + 1 + \mu(\mu - 1) - \beta_2(\mu + 1)]}{3g^2(\beta_2 - 1)^2}, \]

\[ g_3 = \frac{4(\beta_2 + 1 - 2\mu)(2\beta_2 - \mu - 1)(\beta_2 + \mu - 2)}{27g^3(\beta_2 - 1)^3}. \]
The discriminant of \( f(u) \) is

\[
\Delta := g_2^3 - 27g_3^2 = \frac{16(\beta_2 - \mu)^2(\mu - 1)^2}{g_6^6(\beta_2 - 1)^4}.
\]

Hence, \( \Delta \neq 0 \) when \( \mu \neq 1 \) and \( \mu \neq \beta_2 \).

As \( 0 < \alpha_3 < 1 \), we have \( 0 < A := n(n + 1) = 2\alpha_3 < 2 \). Thus

\[
n \in (-2, -1) \cup (0, 1).
\]  

Assume that the system is integrable. Then the identity component of the differential Galois group of the analyzed NVE, i.e., of equation (6.1), is Abelian. Thus, as it is explained in Appendix B, for equation (6.1) we have the Lamé-Hermite, Brioschi-Halphen-Crowford or the Baldassarri case. Now, the Lamé-Hermite case cannot occur because we have restrictions (6.2). We can also exclude the Baldassarri case. In fact, the modular function for the investigated equation is

\[
\text{equation (6.1) we have the Lame-Hermite, Brioschi-Halphen-Crowford or the Baldassarri case. Now,}
\]

As it depends on \( \mu \), by Lemma 14, the Baldassarri case can occur only for finitely many fixed values of \( \mu \), and thus it can be excluded by an appropriate choice of the energy. Taking into account restrictions (6.2), we have the Brioschi-Halphen-Crowford case for equation (6.1) only when \( n = 1/2 \), i.e., for \( \alpha_3 = 3/8 \).

**Remark 3.** In fact Lemma 4 is valid for a wider range of parameters than specified in the lemma. NVE along phase curve \( \Gamma_3^b \) has the form of a Lamé equation when \( \beta_2 \neq 1 \).

To exclude the distinguished value \( \alpha_3 = 3/8 \), we have to investigate all NVEs.

**Lemma 5.** Assume that \( (\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{G}_2^{SG}, \alpha_2 = 1 \) and \( \alpha_3 = 3/8 \). Then the generalised Jacobi problem does not admit any additional meromorphic first integral.

**Proof.** Under the assumptions of the lemma, NVEs along phase curves \( \Gamma_1^b \) and \( \Gamma_2^b \), have the same form as in the generic case analyzed in Section 4, i.e., they have five regular singularities, and the differences of exponents for finite singularities are the same as in the generic case. The differences of exponents at infinity for these NVEs have the forms

\[
\delta_1 = \sqrt{1 + \frac{5(\beta_2 - 1)}{3\beta_2 - 8\beta_3}}, \quad \delta_2 = \sqrt{1 + \frac{5(\beta_2 - 1)}{8\beta_3 - 3}},
\]

so we have

\[
3\delta_1^2 \delta_2^2 - 8(\delta_1^2 + \delta_2^2) + 13 = 0.
\]  

Applying Lemma 1 we deduce that if the identity components of the differential Galois groups of the considered NVEs are Abelian, then \( \delta_1^2, \delta_2^2 \in J \), see (4.6). Notice that for the considered domain of parameters \( \delta_1 \neq 1 \) and \( \delta_2 \neq 1 \).

**Equation (6.3) defines a hyperbola with asymptotes: vertical \( \delta_1 = 8/3 \) and horizontal \( \delta_2 = 8/3 \) provided that \( \delta_1^2 \delta_2^2 \neq 0 \). The maximal values of \( \delta_1^2 \) and \( \delta_2^2 \) are equal to \( \delta_{1, \max}^2 = \delta_{2, \max}^2 = 83/3 \). We can check by direct calculations that for \( \delta_1^2, \delta_2^2 \in [0, 83/3) \cap J \) there is only one pair satisfying this relation namely \( (\delta_1^2, \delta_2^2) = (1, 1) \), but it is excluded by our assumption about the domain of parameters.

Let us assume that \( \delta_1^2 = 0 \). Then relation (6.3) yields \( \delta_2^2 = 13/8 \notin J \). Similarly \( \delta_2^2 = 0 \) implies \( \delta_1^2 = 13/8 \notin J \).

Thus the differential Galois groups of the considered NVEs are \( \text{SL}(2, \mathbb{C}) \), and thus the system is not integrable.
6.2. Sub-case $\alpha_3 = \alpha_2$

When $\alpha_3 = \alpha_2 \neq 1$, then our assumption that $(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{SG}$ implies that

$$\beta_2 \neq \alpha_2 \quad \text{and} \quad \alpha_2 \neq \beta_3 \quad \text{and} \quad \beta_3 \neq \beta_2.$$ 

Now NVE along phase curve $\Gamma^i_1$ for $\mu \notin \{1, \beta_3\}$ has four regular singularities over $\mathbb{C}P^1$

$$z_1 = 0, \quad z_2 = \alpha_2^{-1}, \quad z_3 = \frac{\alpha_2 \mu - \beta_3}{\alpha_2 (\beta_2 - \beta_3)}, \quad z_4 = \infty,$$

and the differences of exponents at these points are following:

$$\Delta_1 = \Delta_2 = \Delta_3 = \frac{1}{2}, \quad \Delta_4 := \delta_1 = \frac{1}{2} \sqrt{1 + \frac{8}{\alpha_2}}.$$ 

Thus this NVE has the form of the Lamé equation. Indeed, when we make the linear change of variable $z = gv + p$, where $g \in \mathbb{C}^*$ is an arbitrary parameter and

$$p = \frac{\beta_2 - 2\beta_3 + \mu \alpha_2}{3(\beta_2 - \beta_3)\alpha_2},$$

then NVE takes the algebraic form of Lamé equation (6.1) with parameters

$$A = \frac{2}{\alpha_2}, \quad B = \frac{2(\beta_2 + \beta_3) - (\mu + 3)\alpha_2}{3g(\beta_2 - \beta_3)\alpha_2^2}.$$ 

Function $f(v) = 4v^3 - g_2v - g_3$ has the invariants

$$g_2 = \frac{4[\beta_2^2 + \beta_3^2 - 2\beta_2\beta_3 - \mu \alpha_2(\beta_2 + \beta_3 - \alpha_2 \mu)]}{3g^2(\beta_2 - \beta_3)^2\alpha_2^2},$$

$$g_3 = \frac{4(\beta_2 + \beta_3 - 2\mu \alpha_2)(2\beta_2 - \beta_3 - \mu \alpha_2)(\beta_2 - 2\beta_3 + \mu \alpha_2)}{27g^3(\beta_2 - \beta_3)^3\alpha_2^3},$$

and the discriminant of $f(v)$ is

$$\Delta = \frac{16(\beta_2 - \mu \alpha_2)^2(\beta_3 - \mu \alpha_2)^2}{g^6(\beta_2 - \beta_3)^4\alpha_2^6}.$$ 

At first we can formulate the following lemma

**Lemma 6.** If for NVE along $\Gamma^i_1$ the identity component of the differential Galois group is Abelian for arbitrary energy values, then

$\hat{\mathfrak{J}} = \{r \in \mathbb{Q} \mid r = p^2/q^2, \quad p \in \mathbb{N}, \quad q \in \{1, 2, 4\}\}.$

**Proof.** Let us write $A = n(n + 1)$ then $\hat{\mathfrak{J}} = [(2n + 1)/2]^2$. The considered NVE is the Lamé equation so the identity component of its differential Galois group is Abelian only in the Lamé-Hermite, Brioschi-Halphen-Crowford or in the Baldassarri case. In the first case $n \in \mathbb{Z}$, in the second case we put $n + 1/2 \in \mathbb{N}$. The last case is excluded because for the considered equation the modular function depends on the energy.

Now, using three NVEs we prove the following.
Lemma 7. Assume that \((\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{SG}\) and \(\alpha_2 = \alpha_3\). Then the generalised Jacobi problem does not admit any additional meromorphic first integral.

Proof. Under the assumptions of the lemma, NVEs along phase curves \(\Gamma_2^n\) and \(\Gamma_3^n\) are the same as in the generic case provided that
\[
\mu \notin \left\{ 1, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_2}, \frac{\alpha_2^2 - \beta_2}{\alpha_2(\alpha_2 - 1)}, \frac{\alpha_2^2 - \beta_3}{\alpha_2(\alpha_2 - 1)} \right\}.
\]

Assume that the system is integrable. Then from Lemmas 1 and 6 it follows that \((\tilde{\delta}_1^2, \tilde{\delta}_2^2, \tilde{\delta}_3^2) \in \tilde{J} \times \tilde{J} \times \tilde{J}\). Furthermore, using the explicit forms of \(\tilde{\delta}_1, \tilde{\delta}_2\) and \(\tilde{\delta}_3\), we find that
\[
4\tilde{\delta}_1^2 \tilde{\delta}_2^2 \tilde{\delta}_3^2 - \tilde{\delta}_2^2 \tilde{\delta}_3^2 - 4(\tilde{\delta}_1^2 + 2\tilde{\delta}_2^2 + 2\tilde{\delta}_3^2) + 17 = 0. \tag{6.4}
\]

From this relation it can be observed that \(0 \leq \tilde{\delta}_1^2 \leq 2\) or \(0 \leq \tilde{\delta}_2^2 \leq 2\) or \(0 \leq \tilde{\delta}_3^2 \leq 2\). In fact, stronger conditions are fulfilled, namely: \(1/4 \leq \tilde{\delta}_1^2 < 16/9\) or \(0 \leq \tilde{\delta}_2^2 < 16/9\) or \(0 \leq \tilde{\delta}_3^2 < 16/9\). Thus either \(\tilde{\delta}_1^2\), or \(\tilde{\delta}_2^2\) or \(\tilde{\delta}_3^2\) belongs to the following set
\[
J_0 = \left\{ 0, 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{4}{16}, \frac{1}{25}, \frac{4}{25}, \frac{9}{25}, \frac{16}{25}, 1, \frac{25}{36}, \frac{25}{36}, \frac{36}{36}, \frac{25}{36}, 49 \right\}.
\]

Next we perform an analysis similar, but much more tedious, as in the proof of Lemma 2. First, we show that equation (6.4) has only a finite number of solutions which belong to \(\tilde{J} \times \tilde{J} \times \tilde{J}\). Then we find them and, finally, we must check if the values of parameters \((\alpha_2, \alpha_3, \beta_2, \beta_3)\) given by these solutions lie in the considered domain \(\mathcal{P}_{SG}\). It appears that there is not a solution satisfying these requirements. \(\blacksquare\)

7. Symmetric non-generic case

In this section we assume that \((\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{SN}\). Let us specify explicitly the sub-cases which should be investigated.

1. When \(\alpha_2 = 1\), then necessarily \(\alpha_3 \neq 1\) and \(\beta_2 \neq 1\). Thus we have two sub-cases:
   (a) \(\beta_3 = \alpha_3\), and
   (b) \(\beta_3 = \alpha_3\beta_2\)

2. When \(\alpha_2 = \alpha_3\), then necessarily \(\alpha_3 \neq 1\) and \(\beta_2 \neq \beta_3\). Hence we have two sub-cases:
   (a) \(\beta_3 = \alpha_3\), and
   (b) \(\beta_2 = \alpha_3\)

We analyse these sub-cases in the following lemmas.

Lemma 8. Assume that \((\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{SN}\), \(\alpha_2 = 1\) and \(\beta_3 = \alpha_3\). Then the generalized Jacobi problem does not admit any additional meromorphic first integral.

Proof. By Remark 3, as \(\beta_2 \neq 1\), conclusions of Lemma 8 are valid for the considered case. Thus, we have to consider only case \(\beta_3 = \alpha_3 = 3/8\). For these values of parameters, NVE along \(\Gamma_2^n\) has three finite regular singularities \(z_1 = 0\), \(z_2 = 8/3\), \(z_3 = 64/15\) with differences of exponents
\[
\Delta_1 = \Delta_2 = \frac{1}{2}, \quad \Delta_3 = \frac{3}{2}.
\]
and order of $z_4 = \infty$ is equal to 3. Assume that the system is integrable. Then the identity component of the differential Galois group of the considered NVE is Abelian. If it is so, then, according to Lemma 13, only Case II of the Kovacic algorithm is possible. The auxiliary sets $E_i$ have the forms

$$E_1 = E_2 = \{1, 2, 3\}, \quad E_3 = \{-1, 2, 5\}, \quad E_4 = \{1\}.$$  

We have only one element $e \in \prod_{i=1}^{4} E_i$ with $d(e) \in \mathbb{N}_0$, namely $e = (1, 1, -1, 1)$ yielding $d(e) = 0$. Equation (9.9) guaranteeing the existence of polynomial $P = 1$ yields condition 2880 = 0. The contradiction shows that the differential Galois group of the considered NVE is SL(2, $\mathbb{C}$). Thus the system is not integrable.

Lemma 9. Assume that $(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{SN}$, $\alpha_2 = 1$ and $\beta_3 = \alpha_3 \beta_2$. Then the generalized Jacobi problem does not admit any additional meromorphic first integral.

Proof. The proof is similar to the proof of the previous lemma but, instead of NVE along phase curve $\Gamma_2^\mu$, we work with NVE along phase curve $\Gamma_1^\mu$.

Lemma 10. Assume that $(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{SN}$ and $\alpha_2 = \alpha_3 = \beta_2$. Then the generalized Jacobi problem does not admit any additional meromorphic first integral.

Proof. Under the given restrictions on values of parameters, NVE along $\Gamma_2^\mu$ has three finite regular singularities $z_1 = 0$, $z_2 = 1$, $z_3 = \alpha_2/(\alpha_2 - 1)$ with differences of exponents

$$\Delta_1 = \Delta_2 = \frac{1}{2}, \quad \Delta_3 = \frac{3}{2},$$

and order of $z_4 = \infty$ is equal to 3. Applying the Kovacic algorithm, we can easily find that the differential Galois group of the considered equation is SL(2, $\mathbb{C}$). Thus the system is not integrable.

Lemma 11. Assume that $(\alpha_2, \alpha_3, \beta_2, \beta_3) \in \mathcal{P}_{SN}$ and $\alpha_2 = \alpha_3 = \beta_3$. Then the generalized Jacobi problem does not admit any additional meromorphic first integral.

Proof. The proof is similar to the previous one, but we should work with NVE along phase curve $\Gamma_2^\mu$.

8. Distinguished energy level

Proving our main theorem we had to investigate three variational equations along three phase curves which lie on the same energy level. Moreover, we always had to avoid specific energy levels. It is natural to ask if the investigated system is integrable on a fixed energy level. Having this in mind, we reanalyzed our proofs. Our observation is following. In a generic case all three NVEs have five singular points. The position of one of these points depends on the energy, and of the other ones depends on parameters $(\alpha_2, \alpha_3, \beta_2, \beta_3)$. For example, for NVE along phase curve $\Gamma_3^\mu$, these points are $z_3$ and $z_4$, respectively. A confluence of these points occurs when

$$\mu = \mu_3^C = \frac{\alpha_2^2 - \beta_2}{\alpha_2(\alpha_2 - 1)}.$$  

Similarly, a confluence of respective points for NVE along phase curve $\Gamma_1^\mu$ occurs for critical value $\mu_1^C$ of $\mu$ given by

$$\mu_1^C = \frac{\alpha_2^2 \beta_3 - \alpha_3^2 \beta_2}{\alpha_2 \alpha_3 (\alpha_2 - \alpha_3)}.$$  

For NVE along phase curve $\Gamma_2^\mu$ this critical value is given by

$$\mu_2^C = \frac{\alpha_2^2 - \beta_3}{\alpha_3 (\alpha_3 - 1)}.$$
For a specific choice of parameters \((\alpha_2, \alpha_3, \beta_2, \beta_3)\) all these three critical energy levels can coincide. A simple analysis shows that if \(\mu_i^C = \mu_j^C\) for \(i \neq j\), then \(\mu_1^C = \mu_2^C = \mu_3^C\), and this occurs when

\[
b_1a_2a_3(a_2 - a_3) + b_2a_3a_1(a_3 - a_1) + b_3a_1a_2(a_1 - a_2) = 0. \tag{8.1}
\]

What is really amazing: for these values of parameters all three normal variational equations given by (3.2) have constant coefficients (i.e., for the specified energy and parameters the Lagrange multiplier (1.3) calculated along a non-stationary solution is constant). Hence, for this case, all three NVEs have trivial differential Galois groups. We considered this fact as a strong indication that the system is integrable on the specified energy level. However, our suspicions appeared to be wrong. We made the Poincaré cross section for a specific choice of parameters satisfying relation (8.1). The results presented in Figure 2 clearly show that the system is not integrable.

![Figure 2](image)

**Fig. 2.** The Poincaré cross section on energy level \(E = 1/2\) for \(a_1 = 1, a_2 = 1/4, a_3 = 1/9, b_1 = -8, b_2 = -5/16\) and \(b_3 = 0\). The cross section plane is \(x_1 = 0\) with coordinates \((x_2, y_2)\).

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9. Appendices

**Appendix A: Linear second order differential equation with rational coefficients and Kovacic algorithm**

Let us consider a linear second order differential equation with rational coefficients

\[
w'' + p(z)w' + q(z)w = 0, \quad p(z), q(z) \in \mathbb{C}(z). \tag{9.1}
\]

A point \(z = c \in \mathbb{C}\) is a singular point of this equation if it is a pole of \(p(z)\) or \(q(z)\). A singular point is a regular singular point if at this point \(\tilde{p}(z) = (z - c)p(z)\) and \(\tilde{q}(z) = (z - c)^2q(z)\) are holomorphic.
An exponent of equation (9.1) at point \( z = c \) is a solution of the indicial equation

\[ \rho(\rho - 1) + p_0\rho + q_0 = 0, \quad p_0 = \tilde{p}(c), \quad q_0 = \tilde{q}(c). \]

After a change of the dependent variable \( z \to 1/z \) equation (9.1) reads

\[ w'' + P(z)w' + Q(z)w = 0, \]

\[ P(z) = -\frac{1}{z^2}p\left(\frac{1}{z}\right) + \frac{2}{z}, \quad Q(z) = \frac{1}{z^3}q\left(\frac{1}{z}\right). \] (9.2)

We say that the point \( z = 1 \) is a singular point for equation (9.1) if \( z = 0 \) is a singular point of equation (9.2). Equation (9.1) is called Fuchsian if all its singular points (including infinity) are regular, see [38], [10].

The Kovacic algorithm [18] allows to decide whether all solutions of equation (9.2) with rational coefficients \( P(z) \) and \( Q(z) \) are Liouvillian. Roughly speaking, Liouvillian functions are obtained from the rational functions by a finite process of solving algebraic equations, integration and taking exponents of integrals. For a formal definition, see e.g., [18].

If one (non-zero) solution \( w_1 \) of equation (9.1) is Liouvillian, then all its solutions are Liouvillian. In fact, the second solution \( w_2 \), linearly independent from \( w_1 \), is given by

\[ w_2 = w_1 \int \frac{1}{w_1} \exp \left[ -\int p \right]. \]

Putting

\[ w = y \exp \left[ -\frac{1}{2} \int p \right] \]

into equation (9.1), we obtain its reduced form

\[ y'' = r(z)y, \quad r(z) = -q(z) + \frac{1}{2}p'(z) + \frac{1}{4}p(z)^2. \] (9.3)

This change of variable does not affect the Liouvillian nature of the solutions. For equation (9.3), its differential Galois group \( \mathcal{G} \) is an algebraic subgroup of \( \text{SL}(2, \mathbb{C}) \). The following lemma describes all possible types of \( \mathcal{G} \) and relates these types to the forms of a solution of (9.3), see [18], [27].

**Lemma 12.** Let \( \mathcal{G} \) be the differential Galois group of equation (9.3). Then one of four cases can occur.

- **Case I** \( \mathcal{G} \) is conjugate to a subgroup of the triangular group

\[ \mathcal{T} = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mid a \in \mathbb{C}^*, b \in \mathbb{C} \right\}; \]

in this case equation (9.3) has an exponential solution of the form \( y = P \exp \int \omega \), where \( P \in \mathbb{C}[z] \) and \( \omega \in \mathbb{C}(z) \),

- **Case II** \( \mathcal{G} \) is conjugate to a subgroup of

\[ \mathcal{G}^l = \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \mid c \in \mathbb{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & c \\ -c^{-1} & 0 \end{bmatrix} \mid c \in \mathbb{C}^* \right\}; \]

in this case equation (9.3) has a solution of the form \( y = \exp \int \omega \), where \( \omega \) is algebraic over \( \mathbb{C}(z) \) of degree 2.
Case III. $\mathcal{G}$ is primitive and finite; in this case all solutions of equation (9.3) are algebraic, thus $y = \exp \int \omega$, where $\omega$ belongs to an algebraic extension of $\mathbb{C}(z)$ of degree $n = 4, 6$ or $12$.

Case IV. $\mathcal{G} = \text{SL}(2, \mathbb{C})$ and equation (9.3) has no Liouvillian solution.

Kovacic in paper [18] formulated the necessary conditions for the respective cases from Lemma 12 to hold.

At first we introduce notation. We write $r(z) \in \mathbb{C}(z)$ in the form

$$ r(z) = \frac{s(z)}{t(z)}, \quad s(z), t(z) \in \mathbb{C}[z], $$

where $s(z)$ and $t(z)$ are relatively prime polynomials and $t(z)$ is monic. The roots of $t(z)$ are poles of $r(z)$. We denote $\Sigma' := \{ c \in \mathbb{C} \mid t(c) = 0 \}$ and $\Sigma := \Sigma' \cup \{ \infty \}$. The order $\text{ord}(c)$ of $c \in \Sigma'$ is equal to the multiplicity of $c$ as a root of $t(z)$, the order of infinity is defined by

$$ \text{ord}(\infty) := \max(0, 4 + \deg s - \deg t). $$

Lemma 13. The necessary conditions for the respective cases in Lemma 12 are the following.

Case I. Every pole of $r$ must have even order or else have order 1. The order of $r$ at $\infty$ must be even or else be lower than 2.

Case II. $r$ must have at least one pole that either has odd order greater than 2 or else has order 2.

Case III. The order of every pole (also at infinity) cannot exceed 2. If the partial fraction expansion of $r$ is

$$ r(z) = \sum_i \frac{a_i}{(z-c_i)^2} + \sum_j \frac{b_j}{z-d_j}, $$

then $\Delta_i = \sqrt{1 + 4a_i} \in \mathbb{Q}$, for each $i$, $\sum_j b_j = 0$ and if

$$ g = \sum_i a_i + \sum_j b_j d_j, $$

then $\sqrt{1 + 4g} \in \mathbb{Q}$.

In [18] Kovacic also formulated a procedure, called now the Kovacic algorithm, which allows to decide if an equation of the form (9.3) possesses a Liouvillian solution and to find it in a constructive way. Applying it, we also obtain information about the differential Galois group of the analyzed equation. Beside the original formulation of this algorithm several its versions and improvements are known [6], [26], [27].

Now we describe the Kovacic algorithm for the respective cases from Lemma 12.

Step 0. We define $\Sigma_i := \{ c \in \Sigma \mid \text{ord}(c) = i \}$ and $\Sigma'_i := \Sigma_i \setminus \{ \infty \}$. For each $c \in \Sigma_1 \cup \Sigma_2$ we calculate the following expansion

$$ r(z) = \frac{a_c}{(z-c)^2} + O \left( \frac{1}{z-c} \right), $$

for $c \in \Sigma_1 \cup \Sigma_2$ and

$$ r = \frac{a_\infty}{z^2} + \frac{b_\infty}{z^3} + O \left( \frac{1}{z^4} \right), $$

for $c = \infty$. 

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We put \( \Delta_c := \sqrt{1 + 4a_c} \).

**Case I**

**Step I.** If \( \text{ord}(\infty) = 0 \), then we define the set \( E_\infty = \{0, 1\} \).

If \( \text{ord}c = 1 \), then we define \( E_c = \{1\} \) for \( c \in \Sigma'_1 \) and \( E_\infty = \{0, 1\} \) for \( c = \infty \).

If \( \text{ord}c = 2 \), then we define for \( c \in \Sigma'_2 \)
\[
E_c := \left\{ \frac{1}{2} (1 + \Delta_c), \frac{1}{2} (1 - \Delta_c) \right\},
\tag{9.4}
\]
and for \( c = \infty \)
\[
E_\infty := \left\{ \frac{1}{2} (1 + \Delta_\infty), \frac{1}{2} (1 - \Delta_\infty) \right\}.
\tag{9.5}
\]

If \( \text{ord}c = 2k \) with \( k \geq 2 \) (only even orders are admissible in this case), then for each \( c \in \Sigma_{2k} \) we compute the rational function \([\sqrt{r}]_c\) defined up to the sign by the following conditions:

- for \( c \in \Sigma'_{2k} \)
\[
[\sqrt{r}]_c = \frac{a_c}{(z - c)^k} + \sum_{2 \leq j \leq k - 1} \frac{s_{j,c}}{(z - c)^j},
\]
\[
r - [\sqrt{r}]_c^2 = \frac{b_c}{(z - c)^{k+1}} + O \left( \frac{1}{(z - c)^k} \right),
\]

- for \( c = \infty \)
\[
[\sqrt{r}]_\infty = a_\infty z^{k-2} + \sum_{0 \leq j \leq k-3} s_{j,\infty} z^j,
\]
\[
r - [\sqrt{r}]_\infty^2 = b_\infty z^{k-3} + O \left( z^{k-4} \right).
\]

Now we define the set \( E_c \) by
\[
E_c := \left\{ \frac{1}{2} \left( \pm \frac{b_c}{a_c} + k \right) \right\}, \quad \text{for } c \in \Sigma'_i,
\]
\[
E_\infty := \left\{ \frac{1}{2} \left( \pm \frac{b_\infty}{a_\infty} - k \right) \right\}.
\]

**Step II.** For each element \( e \) in the Cartesian product
\[
E := E_\infty \times \prod_{c \in \Sigma'} E_c,
\]
we compute
\[
d(e) := e_\infty - \sum_{c \in \Sigma'} e_c.
\]

We select those elements \( e \in E \) for which \( d(e) \) is a non-negative integer. If there are no such elements, equation (9.3) does not have an exponential solution and the algorithm stops here.

**Step III.** For each element \( e \in E \) such that \( d(e) = n \in \mathbb{N}_0 \) we define
\[
\theta(z) = \sum_{c \in \Sigma'} \frac{e_c}{z - c} + \sum_{c \in \Sigma_{2k}, k > 1} s(c) \left[ \sqrt{r} \right]_c,
\]
where \( s(c) \) is + or -, and we search for a monic polynomial \( P = P(z) \) of degree \( n \) satisfying the following equation

\[
P'' + 2\theta(z) P' + (\theta'(z) + \theta(z)^2 - r(z)) P = 0.
\]

If such polynomial exists, then equation (9.3) possesses an exponential solution of the form \( y = P \exp \int \omega \), where \( \omega = \theta \), if not, equation (9.3) does not have an exponential solution.

**Case II**

**Step I.** If \( \text{ord}(\infty) = 0 \), then the set \( E_\infty \) is defined as

\[
E_\infty = \{0, 2, 4\}.
\]

If \( \text{ord} c = 1 \), then we define \( E_c = \{4\} \) for \( c \in \Sigma'_1 \), and \( E_\infty = \{0, 2, 4\} \) for \( c = \infty \).

If \( \text{ord} c = 2 \), then for \( c \in \Sigma'_2 \)

\[
E_c := \{2, 2(1 + \Delta_c), 2(1 - \Delta_c)\} \cap \mathbb{Z},
\]

and for \( c = \infty \)

\[
E_\infty := \{2, 2(1 + \Delta_\infty), 2(1 - \Delta_\infty)\} \cap \mathbb{Z}.
\]

If \( \text{ord} c = k \) and \( k > 2 \), then for \( c \in \Sigma'_2 \), \( E_c = \{k\} \), and \( E_\infty = \{4 - k\} \).

**Step II.** For \( e \in E \) we compute

\[
d(e) := \frac{1}{2} \left( e_\infty - \sum_{c \in \Sigma'} e_c \right).
\]

We select those elements \( e \in E \) for which \( d(e) \) is a non-negative integer. If there are no such elements, Case II cannot occur and the algorithm stops here.

**Step III.** For each element \( e \in E \) such that \( d(e) = n \in \mathbb{N}_0 \) we define

\[
\theta = \theta(z) = \frac{1}{2} \sum_{c \in \Sigma'} \frac{e_c}{z - c},
\]

and we search for a monic polynomial \( P = P(z) \) of degree \( n \) satisfying the following equation

\[
P'' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r) P' + (\theta'' + 3\theta' + \theta^3 - 4\theta\theta' - 2r') P = 0.
\]

If such a polynomial exists, then equation (9.3) possesses a solution of the form \( y = \exp \int \omega \), where

\[
\omega^2 - \psi \omega + \frac{1}{2} \psi' + \frac{1}{2} \psi^2 - r = 0, \quad \psi = \theta + \frac{P'}{P}.
\]

If we do not find such polynomial, then Case II in Lemma 12 cannot occur.

**Case III**

**Step I.** If \( c \in \Sigma' \) has order 1, then \( E_2 = \{12\} \).

If \( \text{ord} c = 2 \), then we define

\[
E_c = \left\{ 6 + \frac{12k}{m} \Delta_c \mid k = 0, \pm 1, \pm 2 \ldots, \pm \frac{m}{2} \right\} \cap \mathbb{Z}.
\]

Here and below in this case \( m \in \{4, 6, 12\} \). For \( c = \infty \) independently of its order we define

\[
E_\infty = \left\{ 6 + \frac{12k}{m} \Delta_\infty \mid k = 0, \pm 1, \pm 2 \ldots, \pm \frac{m}{2} \right\} \cap \mathbb{Z}.
\]

Obviously it can appear that \( a_\infty = 0 \).

**Step II.** For \( e \in E \) we calculate

\[
d(e) := \frac{m}{12} \left( e_\infty - \sum_{c \in \Sigma'} e_c \right).
\]
We select those elements $e \in E$ for which $d(e)$ is a non-negative integer. If there are no such elements, Case II cannot occur and the algorithm stops here.

**Step III.** For each element $e \in E$ yielding $d(e) = n \in \mathbb{N}_0$ we define

$$\theta = \theta(z) = \frac{m}{12} \sum_{c \in \Sigma} \frac{e_c \cdot z - c}{z - c}.$$ 

Next we search for a monic polynomial $P = P(z)$ of degree $n$ satisfying a differential equation of degree $m + 1$ defined in the following way. Put $P_m = P$. Then calculate $P_i$ for $i = m, m-1, \ldots, 0$, according to the following formula

$$P_{i-1} = -SP'_i + [(m-i)S - S \theta]P_i - (m-i)(i+1)S^2rP_{i+1}, \quad (9.12)$$

where

$$S = \prod_{c \in \Sigma'} (x - c).$$

Then $P_{-1} = 0$ gives the desired equation for $P$. If such polynomial exists, then equation (9.3) possesses a solution of the form $y = \exp \int \omega$, where $\omega$ is a solution of the equation

$$\sum_{i=0}^{n} \frac{S^i P_i}{(m-i)!} \omega^i = 0.$$ 

If we do not find such polynomial, then we repeat these calculations for the next element $m \in \{4, 6, 12\}$. If for all $m$ such polynomial does not exist, then Case III in Lemma 12 cannot occur.

**Appendix B: Lamé equation**

The best known is the Weierstrass form of the Lamé equation

$$\frac{d^2y}{dt^2} = (A \varphi(t) + B)y, \quad (9.13)$$

where $A$ and $B$ are, in general, complex parameters and $\varphi(t)$ is the elliptic Weierstrass function with invariants $g_2, g_3$. In other words, $\varphi(t)$ is a solution of the differential equation

$$\varphi^2 = f(v), \quad f(v) = 4v^3 - g_2v - g_3. \quad (9.14)$$

Parameters $A, B, g_2$ and $g_3$ are such that the discriminant $\Delta$ of equation $f(v) = 0$, defined as

$$\Delta = g_2^3 - 27g_3^2,$$

is different from zero. In cases when $\Delta = 0$, $\varphi(t)$ reduces to some simpler functions. The Weierstrass function is an elliptic function with periods $2\omega_1$ and $2\omega_2$, and one pole of a second order localised at $t_0 = 0$ in its fundamental parallelogram. Thus equation (9.13) is Fuchsian on the complex torus $T := \mathbb{C}/L$ where $L = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}$. The exponents at 0 are $-n$ and $n + 1$.

The modular function $j(g_2, g_3)$ associated with the elliptic curve (9.14) is defined as follows

$$j(g_2, g_3) = \frac{g_2^3}{g_2^3 - 27g_3^2}.$$ 

The Fuchsian equation on the Riemann sphere

$$\frac{dy^2}{dx^2} + \frac{f'(x)}{2f(x)} \frac{dy}{dx} - \frac{Ax + B}{f(x)} y = 0, \quad (9.15)$$
where \( f(x) = 4x^3 - g_2x - g_3 \), with parameters \( A, B, g_2 \) and \( g_3 \) such that the discriminant of \( f \) is different from zero, is called the algebraic form of the Lamé equation. The assumption about non-vanishing of the discriminant guarantees that equation \( f(v) = 0 \) has three different roots, which are regular singularities of (9.15). Exponents at these points are 0 and 1/2 and those at infinity \(-n/2\) and \((n + 1)/2\).

The Weierstrass and algebraic forms of the Lamé equation are connected by a variable substitution \( x = \phi(t) \). This transformation is a finite covering and, as a result, the identity components of the Galois groups of equations (9.13) and (9.15) are the same.

Classically, coefficient \( A \) of Lamé equation is written in the form \( A = n(n + 1) \). The values of four parameters \((n, B, g_2, g_3)\) for which the identity component of the differential Galois group of Lamé equation is Abelian correspond to the Lamé-Hermite, Brioschi-Halphen-Crowford and the Baldassarri cases of Lamé equation, see [32], [5]. These are the only cases when the identity component of the differential Galois group of Lamé equation is Abelian. The necessary conditions for the respective cases are listed below.

1. For the Lamé and Hermite case \( n \in \mathbb{Z} \) and three other parameters are arbitrary,
2. For the Brioschi-Halphen-Crowford case \( n + \frac{1}{2} \in \mathbb{N} \) and \( B, g_2, g_3 \) satisfy a certain algebraic equation defined by so-called Brioschi determinant, see e.g. [27].
3. For the Baldassarri case \( n + \frac{1}{2} \in \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z} \setminus \mathbb{Z} \), and there are additional algebraic conditions on \( B, g_2, g_3 \).

Algebraic restrictions on \( g_2, g_3 \), and \( B \) in the Baldassarri case are involved. In applications we use the following lemma which follows from one unpublished result of B. Dwork, see [27].

**Lemma 14.** Assume that the differential Galois group of equation (9.13) is finite. Then for a fixed value of \( A \), the number of possible pairs \((B, j(g_2, g_3))\) is finite.

### References

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