Some abelian varieties admit several (principal) polarizations. The problem of which Jacobi varieties do, especially among the splittable, challenges transliteration from analysis to algebra. Questions and examples are provided, as well as applications to solution of certain Partial Differential Equations.

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1. Introduction

Among the many legacies of Jacobi, the abelian variety that bears his name is of key importance in dynamics. In this brief note, we focus on a difficult problem whose answer is far from known, that of splittable Jacobians, namely those isogenous to a product of lower-dimensional abelian varieties. To create a combination of problems on Jacobi varieties and solutions to the integrable hierarchies (chiefly of KP type, KdV and Boussinesq as a special cases), and to provide some original observations, we focus exclusively on two issues. The first issue is that of reducing, for example, a KP solution to elliptic functions: while there is a classical formula for the theta function, there does not seem to exist one for the constant, which is a function of the moduli of the curve. We revisit an account of parametrization of moduli by quadratic differentials given by A. Tyurin, to interpret the constant. We also propose the (generalized) sigma function as a more natural object than the theta function, to express KP solutions, since it eliminates the constant (the drawback being that there do not yet exist reduction formulas for sigma functions). The second issue is the explicit construction of curves with reduction, and its relation to another challenging concept, that of abelian varieties with many polarizations; in particular, we use elliptic curves with complex, or surfaces with real multiplication. The article’s emphasis is on posing questions, providing a comparative survey of what is known to date, and giving examples to bring out subtle differences.

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The first section below provides foundational material on Jacobian functions and the relevant PDE hierarchies; the second section explores the question of which Jacobians admit several principal polarizations and contains examples and suggested applications. We work over the complex numbers.

2. Elliptic functions, reduction and PDEs

As a link between Jacobi’s work on elliptic functions and his work on dynamics, we take the Korteweg-de Vries equation [57]: $u_t - 6uu_x + u_{xxx} = 0$. In reference to formulas below, we note that by changing the sign of $u$ and taking the variable $t' = -t$, we obtain the version in [48]: $u_t = 6uu_x + u_{xxx}$; similarly, a linear change of variables will be understood when citing formulas that use different versions of the KP equation, which in [8] is given as, $3(u_{yy} - (4u - 6uu_x - u_{xxx})x) = 0$. The general algebro-geometric solution was written by I.M. Krichever in terms of theta functions (1.2) below. We compare the genus-1 version of the relevant function theory ($\sigma$, $\theta$ functions) with higher-genus analogs, with a view to producing explicit solutions in special splitting cases. It is worth to begin by making a philosophical point. The theta function has the advantage of being readily available in $g > 1$, whereas the higher-genus version of the $\sigma$ function has been overlooked until Buchstaber, Enolskii and Leykin [5] began revisiting and extending the work of Baker [2]. Baker and the authors of [5] and subsequent papers, inspired by the algebraic equations of a Burchnall-Chaundy curve, define a $\sigma$ function which for at least three reasons is better suited than the theta function to the theory of completely integrable systems or equations. First we discuss the Burchnall-Chaundy curves; then we address the “three reasons” (our own contention, of course) in turn, and in the course of discussing those we recall the key facts of classical (multi)-periodic function theory.

1.1. Curves with the plane-model property. We say that a curve $X$ (for us, synonymous with compact Riemann surface) has the plane-model property if it has a point $P_\infty$ such that the ring of meromorphic functions on $X$ that are regular outside $P_\infty$, $R := \Gamma(X\setminus\{P_\infty\}, \mathcal{O}_X)$, can be generated by two elements. We then call these functions $x$ and $y$ and view $X$ as an affine curve, which can be completed by the addition of one smooth point. The following fact is easy but we provide a brief justification because we have not been able to find it in the literature.

**Proposition 1.** Not every curve has the plane-model property. In analogy to the numerical constraint on the genus for being a (projective) plane curve, namely $g = \frac{(d-1)(d-2)}{2}$ where $d$ is the degree of the defining (homogeneous) equation, a curve with the plane-model property has genus $\frac{(n-1)(m-1)}{2}$, where $n, m$ are coprime positive integers. A curve has the plane-model property if and only if it is a Burchnall-Chaundy curve, namely the spectral curve of a commutative ring generated by two differential operators of coprime orders $n, m$; in particular, it follows that the equation of the curve can be written as a polynomial $x^m - y^m + (\text{monomials that have order of pole at } P_\infty \text{ less than } n \cdot m)$.

**Proof.** Assume first that $X$ has the plane-model property and $x, y$ are the generating functions; let $n, m$ be the order of pole of $x, y$ at $P_\infty$ (respectively). By the Weierstrass’ gap theorem (or more simply, an application of Riemann-Roch), $\dim H^0(\mathcal{O}((N+1)P_\infty)) - \dim H^0(\mathcal{O}(NP_\infty)) = 1$ when $N \gg 0$, and these spaces are spanned by monomials in $x, y$, so $n$ and $m$ must be coprime. Finding the genus, again by Riemann-Roch, is now a matter of a “postage-stamp problem” (if there are two stamp values $n$ and $m$, what is the largest amount that cannot be paid exactly?) and an elegant proof is provided by Burchnall and Chaundy themselves [6]. If the curve is non-hyperelliptic, the lowest of the two values cannot be 2, and it’s easy to see that under such conditions $\frac{(n-1)(m-1)}{2}$ cannot be 5. By viewing the inverse $z$ of a local parameter at $P_\infty$ as spectral parameter, and using a Baker-Akhiezer (eigen-)function, we can send $R$ isomorphically to a ring of differential operators in such a way that the order is the same as the order of pole at $P_\infty$; the equation of the curve is the determinant of the Burchnall-Chaundy matrix. The same construction gives the converse.
The authors of [5] call these \((n,m)\) curves. It is possible in principle, difficult to put in close form and so far completed for the \(n = 2\) (hyperelliptic) case, to use a convenient basis of differentials of first and second kind to obtain an analog of the \(\sigma\) function. We discuss “three reasons” for its importance.

One reason is that in the rational limit, the function \(\sigma\) approaches Schur functions [48], which are polynomial in their variables; useful information on the multiplicities of zeros or poles ensues.

A second reason is a relevance to the HBDE (Hirota Bilinear Difference Equation), which holds for Sato’s \(\tau\)-function (indeed related to Schur functions; cf. [56]):

\[
(\nu_2 - \nu_3)\tau(\ell - \epsilon[\nu_1])\tau(\ell - \epsilon[\nu_2]) + (\nu_3 - \nu_1)\tau(\ell - \epsilon[\nu_3])\tau(\ell - \epsilon[\nu_2]) - (\nu_1 - \nu_2)\tau(\ell - \epsilon[\nu_3]) + (\nu_1 - \nu_3)\tau(\ell - \epsilon[\nu_1])\tau(\ell - \epsilon[\nu_2]) = 0
\]

where \(\nu_1, \nu_2, \nu_3\) are three complex numbers and the “shift” by a number \(\lambda\) is defined as the infinite sequence:

\[
\epsilon[\lambda] = \left\{ \frac{1}{\lambda}, \frac{1}{2\lambda^2}, \frac{1}{3\lambda^3}, \ldots \right\}.
\]

It is remarkable that this 3-term relation implies the whole KP hierarchy. Compare with the “three term equation”

\[
\sigma(z + a)\sigma(z - a)\sigma(b + c)\sigma(b - c) + (z + b)\sigma(z - b)\sigma(a + c)\sigma(c - a) + (z + c)\sigma(z - c)\sigma(a + b)\sigma(a - b) = 0
\]

which characterizes the classical \(\sigma\) function (see Ref. [55]-Examples 5, 38, due to Hermite),

\[
\sigma(z) = z \prod_{(m,n) \neq (0,0)} \left( 1 - \frac{z}{m\omega_1 + n\omega_2} \right) \exp \left( \frac{z}{m\omega_1 + n\omega_2} + \frac{z^2}{2(m\omega_1 + n\omega_2)^2} \right).
\]

To bring into play the other relevant functions, we recall that the function \(\sigma\) is defined so that

\[
\frac{d^2}{dz^2} \log \sigma = -\vartheta, \quad \text{and} \quad \lim_{z \to 0} \frac{\sigma(z)}{z} = 1,
\]

in analogy with the function \(\sin z\), so that \(\frac{d}{dz} \log \sigma(z) = \zeta(z)\) in analogy with \(\frac{d}{dz} \log \sin z\). In this formula, the theta function with characteristics is:

\[
\vartheta[\alpha](z) := \sum_{m \in \mathbb{Z}^g} \exp \left( \frac{1}{2}(m + \alpha_1)\Omega^f(m + \alpha_1) + (z + 2\pi i\alpha_2)^f(m + \alpha_1) \right), \tag{2.1}
\]

in Fay’s normalization for the period lattice, namely the matrix \([2\pi i, \Omega]\) and the characteristic \(\alpha = \alpha_1 + 2\pi i + \alpha_2\Omega, \alpha_1, \alpha_2, z \in \mathbb{C}^g\); the “prime form” \(E(x,y)\) is a multiplicative \((-\frac{1}{2})\)-order differential in \(y\). Our contention for the preferred role of the (higher-genus) \(\sigma\) function over \(\vartheta\) is based on the greater symmetry of this crucial formula when expressed in terms of \(\sigma\).

The third reason is the one that has most implications. In terms of the theta function, the solution to the KP hierarchy,

\[
u(t) = -2\frac{\partial^2}{\partial t^2} \log \vartheta \left( \sum_{i \geq 1} t_i U_i + U_0 \right) + \text{const}, \tag{2.2}
\]

where \(U_i \in \mathbb{C}^g\) are suitable vectors (see Ref. [57]-Ap. 6.4) involves a constant that depends on the curve \(X\). It is the significance of this constant that brings out the difference between \(\sigma\) and \(\vartheta\). To show this, firstly, we recall the calculation of the constant using the Baker-Akhiezer function; secondly, we interpret it as a projective connection; thirdly, we give examples (genus 1, and hyperelliptic cases); finally, we suggest an algorithm for the case when the Jacobian has reduction, to express the constant in terms of elliptic functions.
Calculation. The KP hierarchy is the sequence of time-deformations for a (formal) pseudo-differential operator: $L = \partial + u_{-1}(\tilde{t}) \partial^{-1} + \ldots \in \Psi = \{ \sum_{n=0}^{\infty} u_{j}(\tilde{t}) \partial^j, \partial := \frac{\partial}{\partial t_1}, u_j \text{ analytic near } t_1 = 0, n \in \mathbb{Z} \}$, where $t_j$, $j > 0$, play the role of parameters satisfying
\[
\partial_t \mathcal{L} = [(\mathcal{L}^j)_+, \mathcal{L}]
\]
where $( )_+$ is projection $\Psi \to \mathcal{D}$. These flows translate into a set of PDE’s on $u_1(\tilde{t})$. A solution $L$ is “stationary” with respect to $t_j$ if $\mathcal{L}^j \in \mathcal{D} = \{ \sum_{k=0}^{\infty} c_j(\tilde{t}) \partial^k \} \subset \Psi$ (which implies $\partial_t \mathcal{L} = 0$), e.g. for $j = 2$ we get KdV. More generally, let $K_j = (\mathcal{L}^j)_+$ and say that a KP solution is stationary if a nontrivial combination $\sum_{i=1}^{N} c_j \mathcal{L}^i \in \mathcal{D}$, i.e. the corresponding time operator $\sum_{i}^{N} c_j K_j$ acts trivially.

When the centralizer $C_D(L) = \{ \sum_{n-\infty} c_j \mathcal{L}^i, c_j \in \mathbb{C} \} \cap \mathcal{D}$ is a ring that contains an operator of any sufficiently large order $N$, then it can be viewed as the ring of meromorphic functions on a Riemann surface $X$, regular outside a point $P_\infty$; the order of the operator corresponds to the order of pole of the function. The common eigenfunction of the ring is known as Baker-Akhiezer function:
\[
\psi(\tilde{t}, P) = \exp \left( \sum_{i} t_i \int_{0}^{P} \eta_i - \sum_{i} t_i b_{i0} \right) \frac{\vartheta(A(P) + \sum_{i} t_i U_i - U_0) \vartheta(A(P) - U_0) \vartheta(A(P) + \sum_{i} t_i U_i - U_0)}{\vartheta(A(P) - U_0) \vartheta(A(P) + \sum_{i} t_i U_i - U_0)}
\]
where $\vartheta(-)$ is the Abel map with base point $P_0$, $U_0$ is given in terms of the Riemann constant, which normalizes the theta divisor, and a non-special divisor on the curve, $U_i \in \mathbb{C}$ are the time flows, $\eta_i$ are regular outside a point $P_\infty$; the order of the operator corresponds to the order of pole of the function. The common eigenfunction of the ring is known as Baker-Akhiezer function:
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where $\vartheta(-)$ is the Abel map with base point $P_0$, $U_0$ is given in terms of the Riemann constant, which normalizes the theta divisor, and a non-special divisor on the curve, $U_i \in \mathbb{C}$ are the time flows, $\eta_i$ are normalized differentials of the second kind expanded at $P_\infty$ in terms of a local parameter, the inverse of the spectral parameter $z$:
\[
\int_{0}^{P} \eta_i = z^i + \sum_{k=0}^{\infty} b_{ik} z^{-k}.
\]
Using the periodicity of $\vartheta$ and the Krichever map, see [48]-§6, §9, the constant in the solution above can be calculated to be $-2h_{11}$. By linear or Galilean change of variables, Dubrovin [7]-(3.2.1) normalizes the constant, so that the for the solution above (for brevity we now single out the variable $x := t_1$ and understand $\tilde{t} = (t_2, \ldots)$):
\[
u(x, \tilde{t}) = -2 \frac{\partial^2}{\partial x^2} \log \vartheta \left( u_1 x + u_2 t_2 + (u_3 + \text{const.}) t_3 \right)
\]
and computes the constant [8]-1.7, const $= c(P_\infty)$ where:
\[
3c(P) = \sum \frac{\omega''_0(P) \vartheta_1(\zeta)}{\sum \omega_0(P) \vartheta_1(\zeta)} \frac{3}{2} \left( \sum \frac{\omega'_0(P) \vartheta_1(\zeta)}{\sum \omega_0(P) \vartheta_1(\zeta)} \right)^2 + \frac{3}{2} \frac{\vartheta_x(\zeta)}{\vartheta(\zeta)}^2 - \frac{\vartheta_{xx}(\zeta)}{\vartheta(\zeta)}
\]
where $\omega_i$ are the normalized differentials of the first kind and $\zeta$ is any nonsingular point on the theta divisor. This projective connection is independent of $\zeta$ and also appears in [12]-p. 19, Eq. 27. Cf. [44]-Ap. for a cohomological interpretation of $c(P)$.

A projective connection. The beautiful introductory paper [50] justifies the formula above by explaining the significance of a “projective connection”, also called a projective structure, and its relationship with quadratic differentials. Both, roughly speaking, encode the moduli, or holomorphic structure, of the Riemann surface. Following [50]:

1.2. Definitions and facts. (i) A coordinate system $\{ (U_i, z_i) \}$ on a Riemann surface is called a flat structure if the transition functions $\alpha_{ij}$ are fractional linear transformations. (ii) The Schwarzian derivative of a function with respect to a given variable, commonly denoted by $\{ f, z \}$, is the following third-order differential operator:
\[
S_z^{f} := \frac{f'''(z)}{f''(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]
The notation is convenient because the symmetric expression

\[ S^{p,q}_{x,y} := \frac{f'(x)f'(y)}{(f(x) - f(y))^2} - \frac{1}{(x - y)^2}, \quad p = f(x), \quad q = f(y) \]

reduces to \( \frac{1}{6}S^f_i \) when \( x = y = z \), and it is motivated by the expansion:

\[
\log\left(\frac{f(x) - f(y)}{x - y}\right) = \log f'(z) + \frac{1}{2} f''(z)(x_1 + y_1) + \frac{1}{6} \left[ \frac{f'''(z)}{f'(z)} - \frac{3}{4} \left( \frac{f''(z)}{f'(z)} \right)^2 \right] (x_1^2 + y_1^2) - \frac{1}{6} S^f_i x_1 y_1 + \ldots
\]

for the variables \( x_1 = x - z \), \( y_1 = y - z \). Note

\[
S^{p,q}_{x,y} = \frac{\partial^2 \log(f(x) - f(y))}{\partial x \partial y}.
\]

(iii) The Schwarzian derivative is invariant under fractional linear transformations of \( f \), \( S^{(\alpha f + \beta)/(\gamma f + \delta)}_i = S^f_j \). (iv) Under a fractional linear transformation of the argument the Schwarzian derivative transforms as a quadratic differential: \( S^{(\alpha f + \beta)/(\gamma f + \delta)}_i \). (v) For \( f \) a function of \( z \), \( f \) is a solution of the third-order differential equation: \( S^f_i = q(z) \) if and only if \( f = y_1/y_2 \), where the \( y_i \) are a pair of independent solutions of the second-order differential equation: \( y'' + \frac{2}{y} = 0 \). This implies a converse to (iii), namely: (vi) If \( S^f_i \equiv 0 \), then \( f = \frac{\alpha z + \beta}{z + \gamma} \). (vii) A collection of (regular) functions \( \{h_j\} \) is called a (regular) projective connection for the covering \( \{(U_i, z_i)\} \) if on each intersection \( S^{h_j}_{z_i} = h_j \left( \frac{\partial x}{\partial z_i} \right)^2 - h_j \). For any two projective connections \( \{h_i\} \) and \( \{h'_i\} \), their difference is a regular quadratic differential \( \omega \) on \( X \). If \( z_i \) are flat coordinates, then a projective connection is a regular quadratic differential; conversely, a projective connection defines a flat structure on \( X \), as follows: for any solution \( g_i \) of the differential equation: \( S^{g_i}_{z_i} = -h_i \), the new coordinates \( z_i^h = g_i(z_i) \) are easily checked to define the same complex structure on \( X \), and to be flat. (viii) The set of flat structures on a Riemann surface \( X \) is in bijection with the set of regular projective connections, and the moduli space \( M \) of flat structures on \( X \) is affinely isomorphic to the space of quadratic differentials on \( X \), \( H^0(X, O(2K_X)) \). (ix) A projective connection can be specified by ‘restriction to the diagonal’: Define a tensor \( \omega(x, y) \) on \( X \times X \) to be a symmetric bidifferential of the second kind if \( \omega(x, y_0) \) is a differential on \( X \) with a single pole of order 2 for any fixed \( y_0 \in X \) and \( \omega(x, y) = \omega(y, x) \) (equivalently, \( \omega(x, y) \) is a symmetric section of the sheaf \( (p_1^*O \otimes p_2^*O)(2\Delta) \), where \( p_i \) are the two projections from the cartesian product, \( \Delta \) is the diagonal and \( O \) is the canonical sheaf on \( X \) ). The biresidue \( \text{Bires}(\omega) \) is the number \( a \) locally defined by:

\[ \omega = \frac{\partial xy}{(x-y)^2} + H(x, y) dx dy, \]

where \( H dx dy \) is an entire bidifferential and \( x, y \) on the two factors are the same local coordinate: \( \text{Bires} \) is independent of coordinates. If \( \omega \) is a symmetric bidifferential of the second kind and \( \text{Bires}(\omega) = 1 \), the function \( h_\omega(z) := -6H(x, y) \big|_{x=y=z} \) is a projective connection.

(x) There is a unique symmetric bidifferential of the second kind, \( \omega_{A,B} \), normalized with respect to a standard symplectic basis \( \{A_i, B_j\} \) of the first homology of \( X \): \( \text{Bires}(\omega_{A,B}) = 1 \); \( \int_{A_i} \omega_{A,B}(x, y) = 0 \) for any fixed \( y \in X \; \int_{B_j} \omega_{A,B}(x, y) = \omega_i(x) \), a normalized basis of \( H^0(X, O) \). This differential is called the Bergman kernel. (xi) For any even \( \theta \)-characteristic \( \theta \) with non-zero thetanull,

\[
\sum_{i,j=1}^{g} \omega_i(x) \omega_j(y) \frac{\partial^2}{\partial z_i \partial z_j} \log \theta \bigg|_{z=0} \]

is a holomorphic bidifferential on \( X \times X \) and \( \omega_{[1]} := \omega_{A,B} + \sum_{i,j=1}^{g} \omega_i(x) \omega_j(y) \frac{\partial^2}{\partial z_i \partial z_j} \log \theta \bigg|_{z=0} \) is invariant under a change of homology basis preserving the theta characteristic; this symmetric invariant bidifferential of the second kind is called the Klein bidifferential, and its associated projective
connection is called the Wirtinger connection $w_{[n]}$. This is the constant we encountered in the KP solution.

Our emphasis on the $\sigma$ function over the Riemann theta function as a solution, not only to the HBDE as sketched above, but to the KP hierarchy, differing from the approach of [48]-3-9 where $\tau$ is compared to $\vartheta$, is vindicated by the following:

1.4. Example: $g=1$. Translation by periods gives rise to certain invariants of the Riemann surface: $\varphi(z + 2\omega) = \varphi(z)$, $\zeta(z + 2\omega) = \zeta(z) + 2\eta_1$, $\zeta(z + 2\omega_2) = \zeta(z) + 2\eta_2$, $\eta_i = \zeta(\omega_i)$, $\sigma(z + 2\omega_1) = -e^{2\pi i (z + \omega_1)} \sigma(z)$, $\sigma(z + 2\omega_2) = -e^{2\pi i (z + \omega_2)} \sigma(z)$. Note: $\eta_1\omega_2 - \eta_2\omega_1 = \frac{1}{2}\pi i$.

(i) [55]-21-43 gives the “connexion of the Sigma-function with the Theta function”:

$$\sigma(z) = \frac{2\omega_1}{\pi} \exp\left(\frac{\eta_1 \pi^2}{2\omega_1} \frac{1}{2} q^{-1/4} \prod_{n=1}^{\infty} (1 - q^{2n})^{-3} \frac{\pi z}{\omega_2} \frac{1}{\omega_1}\right),$$

where $q = e^{\pi iz}$, $\tau = \omega_2/\omega_1$, and

$$\vartheta_4(z, \tau) = -ie^{\pi i z + \frac{1}{4}\pi i \tau} \vartheta_4\left(z + \frac{1}{2}\pi \tau, \tau\right) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)iz}.$$

In turn, $\vartheta_4$ is the function that Jacobi denoted by $\vartheta(\pi z)$ [55]-21-9 and $\vartheta_4(\pi (z + \frac{1}{2}))$ is Riemann’s theta function [22]-1.10

$$\vartheta_4(z, \tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^n e^{2niz}.$$

(ii) The link between $\varphi$ and the Schwarzian derivative is given in [55]-21-43:

$$\varphi(z) = -\frac{\eta_1}{\omega_1} + \left(\frac{\pi}{2\omega_1}\right)^2 \csc^2\left(\frac{\pi z}{2\omega_1}\right) + \left(\frac{\pi}{2\omega_1}\right)^2 \left[ \frac{\vartheta''(\nu)}{\vartheta(\nu)} - \left(\frac{\vartheta'\nu}{\vartheta(\nu)}\right)^2 \right]$$

where $\nu = \frac{1}{2} \frac{\pi z}{\omega_1}$ and $\vartheta_1 = \sin \cdot \varphi(z)$, giving

$$\sigma(z) = \frac{2\omega_1}{2\vartheta_1(0)} \exp\left(-\frac{\vartheta''(0)}{6\vartheta_1(0)}\right) \vartheta_1(\nu)$$

and ultimately, recalling the Dubrovin normalization [8]-1.7 for the genus 1 (KdV, with $t = t_3$) case: $u(x,t) = -2\partial_x^2 \log \vartheta(u_1 x + (u_3 + \frac{\omega_3}{6})t)$, where $u_1 = \frac{1}{\omega_1 - \omega_2}$ to normalize: $\int_{e^2} \omega = 1$, $\int_{e^3} \omega = 0$, $\omega = \frac{1}{\omega_1 - \omega_2} \frac{d(x-e_2)}{y}$, the solution is indeed given by $\sigma$ without a constant shift:

$$\frac{d^2}{dz^2} \log \sigma(z) = \frac{2\omega_1}{2\vartheta_1} \left(1 + \frac{1}{12} \frac{\vartheta''(0)}{\vartheta_1} + \frac{d^2}{dz^2} \log \vartheta_1(\nu) \right).$$

A similar normalization of the shift occurs, of course, at the level of the Bergman kernel $\omega = [\varphi(x - y) - \eta] dx dy$, [50]-1.3 where $\eta = \frac{1}{3} \vartheta''(\frac{1}{3})$ and the averaged Klein bidifferential, which becomes an invariant of the Riemann surface, $\omega_X = \omega_{A,B} + \frac{1}{3} \vartheta''(\frac{1}{3}) dxdy = \varphi(x-y) dxdy$

(1)\] (as noted in [50], this says that the invariant Wirtinger connection $w_{[n]}$ gives the unique flat additive coordinate on the torus).

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Hyperelliptic example. When $X$ is hyperelliptic and $P_{\infty}$ is a Weierstrass point, by using the square root of a local parameter as the inverse of the spectral parameter $z$, as we saw the KP equation reduces to KdV ($t_2$-independent). In this case the constant has a more explicit interpretation. As recalled in [57], using the spectral theory of a finite-gap Hill’s operator with gap edges $e_0, \ldots, e_{2g}$, and the Floquet eigenfunctions, the KdV solution: $u(x,t) = -\frac{d^2}{dx^2} \log \vartheta(xU + tV + K) + \text{const}$ has
\[ \text{const} = \sum_{i=0}^{2g} e_i - 2 \sum_{i=0}^{g} \frac{\vartheta_{i}^{g}}{\vartheta_{i}^{g+1}} \text{ [57]-II (9.11)} \] – the normalization of the differentials of the first kind is the same as in [12]. Via the theory of Hill’s equation, a renormalization can also be written as average of a spectrum: in [15] the most general spectrum in question, $\lambda_{0}^{g}, \ldots, \lambda_{g}^{g}$, is analyzed, interpolating boundary conditions for a (normalized) eigenfunction $g(x)$ of $L(t) = -\frac{d^2}{dx^2} + u(x)$ (we omit the $t$-dependence in this notation since it is isospectral when KdV is satisfied). The condition is: $g'(x) + \ell g(x) = 0, \ell \in \mathbb{R}$, the case of $\ell = \infty$ and $\ell = 0$ being respectively the Dirichlet and Neumann spectra (the KdV dynamics is reduced to a system of ODEs for the Dirichlet spectrum); then trace formulas give:
\[ 2\ell^2 - u = \sum_{i=0}^{2g} e_i - 2 \sum_{i=0}^{g} \lambda_i^{g} \text{ [15]-I.85} \text{ and theta-function theory gives:} \]
\[ u(x) = \sum_{i=0}^{2g} e_i - 2 \sum_{i=1}^{g} \lambda_i - 2\partial^2 \log \vartheta^{2} \text{ [15]-I.108}. \]
However, the $\lambda_{1}, \ldots, \lambda_{g}$ are the zeros of a normalized differential of the second kind and do not depend on $\ell$; thus the use of $\vartheta$ appears to have introduced an invariant of the Riemann surface that does not have a natural spectral interpretation; perhaps, by considering limits of finite-gap invariants and letting the number of gaps go to infinity, such an interpretation could be found via the inverse-spectral techniques of [36]. At the same time, projective connections have an explicit expression in terms of the branchpoints: if the curve is given by $X : y^{2} = \prod_{i=1}^{2g+2} (x - \alpha_i)$, $h_{j} := S_{x}^{j} + \frac{3}{8} \left( \frac{d}{dx} (\log \prod_{i=0}^{g} \frac{x(z_{i}) - \alpha_{2i+1}}{x(z_{i}) - \alpha_{2i+2}}) \right)^{2} (z_{j}$ local coordinates) is a projective connection [50]-I.3., Exercise 2 and can be shifted by $-\sum_{j,k}^{g} \frac{\partial^{2} \log \vartheta^{j}}{\partial \vartheta_{j} \partial \vartheta_{k}} (0) \omega_{j}(z_{i}) \omega_{k}(z_{i})$ (the $\zeta_{j}$ are the abelian coordinates associated to the basis $\omega_{j}$) to become a Wirtinger connection [12]-II.

To close the section, we introduce the question of reduction. When holomorphic differentials on $X$ can be expressed in terms of elliptic differentials, an effective solution to the KP hierarchy is sought.

Algorithm. If Jac$X$ is isogenous to a product of elliptic curves, the theta function can be expressed in terms of elliptic functions by using classical formulas (cf. [38]), provided the period matrix is explicitly given. Using the theta-function expression given above, we can also ‘reduce’ the expression for the constant. It may be interesting to find an algebraic algorithm for reduction, in the hyperelliptic case, given the Weierstrass points of $X$ and the elliptic curves in Weierstrass form: $(x')^{2} = 4y^{3} - g_{2}y - g_{3}$. I do not know of an explicit way to reduce the higer-genus generalization of $\sigma$ to an expression of classical ones, despite the comprehensive programme beautifully laid out in [3], [4].

3. Ppavs with many principal polarizations; split Jacobians

In view of the importance of a splitting as regards applications to solutions of PDEs, we switch to the theoretical question of abelian varieties which contain an elliptic curve. It will be apparent below that so little is known, that we feel justified in limiting ourselves to dimension 2 or 3 in this short note. Another restriction that pertains to the dynamics applications is that the abelian variety be “principally polarized”, a papv for short, or a ppas in the 2-dimensional case (a surface). There are many definitions of polarization, so it seems wise to recall one, of specific use to us. A principal polarization of an abelian variety $A$ may be thought of as an ample line bundle $\Theta$ on A such that the homomorphism $A \rightarrow \text{Pic}^{0}(A)$ given by $a \mapsto (T_{*} a \Theta) \otimes \Theta^{-1}$ is an isomorphism. This definition is particularly useful because it immediately gives the isomorphism between the ppav and its dual $\text{Pic}^{0}A$, which will be used below. It is known [41] that the set of isomorphism classes of principal polarizations of $A$ is finite: this seems to settle the question posed in [37].

There is much activity in trying to identify curves whose Jacobians are isomorphic. By Torelli’s
Theorem, this implies that they have several principal polarizations. There can occur three situations that have bearing on the KP dynamics: (I) \( X \) could admit an “elliptic subcover”, namely a nonconstant map \( \pi : X \to E \) to an elliptic curve; (II) \( \text{Jac}X \) could split into a product of elliptic curves, by isogeny or by isomorphism; (III) In situation (I), \( \pi \) could be a “tangential cover”. This concept was introduced by Treibich and Verdier ([52], [54]) to classify KP solutions elliptic (doubly-periodic) in the first variable, which they called “elliptic solitons”. Referring to [54] for details, we recall that a (pointed) cover \( \pi : (X, p) \to (E, q) \) is tangential when the natural images of the two curves in \( \text{Jac}X \) (with any normalization of the Abel map) are tangent at the image of the points \( p, q \).

After preliminary observations, we highlight some of what’s known, then provide (non-)examples.

To relate situations (I) and (II), one observation which is not hard to prove by the use of the natural self-duality of a ppav, but is very important, is that the subcover is minimal if and only if the elliptic curve is contained in the ppav:

**Proposition 2.** An elliptic subcover is said to be minimal when it does not factor through an isogeny. If \( \pi : X \to E \) is a minimal subcover, then the kernel of \( \pi^* : E \to \text{Jac}X \) is trivial, so \( E \) can be translated into a subgroup. Conversely, any elliptic subgroup of \( \text{Jac}X \) is a minimal subcover. In this situation, the degree of the subgroup, namely its intersection number with a principal polarization, equals the degree of the cover.

**Proof.** This is proved in [23]-Prop. 4.2 (Cor. 4.3) by universal properties. It can be seen directly, by self-duality, because the transpose of an injective map is a projection. The assertion on the degree follows from group-variety properties.

**Situation (I)** in moduli is explored in [1], chiefly in dimension 2. Four approaches are compared: modular (analytic); algebraic (via equations); group-theoretic (via Galois or monodromy groups); and topological (via covering spaces). One aspect that is not mentioned in [1], and we briefly illustrate here because it provides the richest source of examples, is that of curves with complex or real multiplication. However, we need some preliminary results that pertain to problem (II). We make use of some key, nontrivial, facts:

- An abelian variety \( A \) is said to be “exceptional” if the rank of its Néron-Severi group has the maximal possible value. In characteristic 0, this value is \( g^2 \) and the following three properties are equivalent: (i) \( A \) is exceptional; (ii) \( A \) is isogenous to a product of \( g \) mutually isogenous singular elliptic curves; (iii) \( A \) is isomorphic to a product of \( g \) mutually isogenous singular elliptic curves [30]-[31]. (Results of a similar nature were proved around the same time by other authors; we only cite the one that is most informative for our purposes and apologize for any omission.)

- In dimension 2, any ppas is a Jacobian, provided it is not reducible. With this in mind, the leading question in [14] is: when is the product of two elliptic curves isogenous to a Jacobian? The answer is keyed into an anti-isometry (with respect to the Weyl pairing) between the subgroups of points of period \( d \) in each of the two curves, and the number of the corresponding Jacobians is computed in [24]-[25]. In the case that the elliptic curves \( E \) and \( E' \) are not isogenous, the surface \( E \times E' \) is always isogenous to a Jacobian; but in the case when they are isogenous, a difficult numerical invariant must be computed.

**Situation (II).** There are classes of examples given by curves with CM. Recall that an elliptic curve is said to have complex multiplication (CM) when \( \text{End}(E) \) contains properly \( \mathbb{Z} \). In characteristic 0, the only possibility for such an \( \text{End}(E) \) is to be an order \( R \) in an imaginary quadratic field \( K \), which means \( R \otimes \mathbb{Q} = K \), cf. [49].

- If a product of an elliptic curve \( E \) by itself admits a non-reducible polarization (in particular, is a Jacobian) then \( E \) must have CM [37].

In [18], the result is the following: Let \( E \) and \( E' \) be elliptic curves whose endomorphism rings are both isomorphic to the principal order of an imaginary quadratic field \( \mathbb{Q}[\sqrt{m}] \). Assume there exist endomorphisms \( \lambda_1, \ldots, \lambda_r \) of \( E \) such that \( E' \) is isomorphic to the diagonal image of \( E \) under the product of these endomorphisms. Then, \( E \times E' \) is the Jacobian of a genus-2 curve if and only if
When the curve \( E \) does not have CM, [16] shows that the product of \( E \) with an isogenous curve \( E' \) can be the Jacobian of a divisor, and the number of isomorphism classes is an integral multiple of the class number of \( \mathbb{Q}(\sqrt{-m}) \), where \( m \) is the minimal degree of a homomorphism from \( E \) to \( E' \). In particular, the formulas imply, by Siegel’s theorem, that there are only a finite number of which \( E \times E' \) is not a Jacobian. The precise result is that \( E \times E' \) is not a Jacobian if and only if \( m \) is one of the following: \( m = 1, 4, 12 \); or, \( m = f^2m_0 \) (\( m_0 \) square-free), every ideal class is ambiguous, and: \( f = 1 \), \( m_0 \equiv 2 \mod 4 \); \( f = 3 \), \( m_0 \equiv 2 \mod 12 \); \( f = 2 \), \( m_0 \equiv -1 \mod 8 \); or \( f = 6 \), \( m_0 \equiv -1 \mod 24 \). This should be revisited in the light of [14]’s criterion, although in [14] the Jacobians are in general only isogenous to products. Moreover, in [19] the authors explicitly describe the endomorphism ring of the resulting Jacobian, a question that comes up next.

In [40], examples are given of hyperelliptic Jacobians with real multiplication, associated to any isogeny \( E \to E' \) of elliptic curves; for the special case of genus 2, the real multiplication is by the number \( \frac{1 + \sqrt{2}}{2} \), and this example is classical (Humbert). Moreover, Mestre gives explicit equations for a 2-parameter family of such Jacobians. On the elliptic curve \( E : y^2 + (1 - U)xy - Uy = x^3 - Ux^2 \) the point \( R = (0, 0) \) has order 5. The equation of the isogenous curve \( F := E/\langle R \rangle \) is given explicitly in [39]. The curve of genus 2 \( X : Y^2 = (1 - Z)^3 + UZ((1 - Z)^3 + UZ^2 - Z^3(1 - Z)) - TZ^2(Z - 1)^2 \) is the same as the double-cover of \( \mathbb{P}^1 \) that is branched where \( T = (\text{the image of a point of } E/\langle R \rangle \text{ in } \mathbb{P}^1 \text{ under the } \pm \text{ identification}) \); in particular it is a 5 : 1 cover of \( E/\langle R \rangle \). The Jacobian \( \text{Jac}X \) is isogenous, hence as we reviewed, isomorphic, to a product \( (E \times E)/(\mathbb{Z}/5\mathbb{Z}) \), and the curve \( X \) can be constructed by using a theorem of Poncelet, roughly as follows: the quotient of \( X \) by the hyperelliptic involution can be viewed as a plane cubic; by ordering the Weierstrass points, \( \{P, P_1, \ldots, P_5\} \), one sets up a bijection between such curves \( X \) and conics inscribed in the pentagon \( \{P_1, \ldots, P_5\} \), which is in turn recording a pair of data: elliptic curve, and point of order 5 on it. Perhaps a Poncelet’s theorem in space [43] can be used to construct covers of hyperelliptic curves, with application to dynamics by reduction to hyperelliptic theta functions.

In dimension 2, there is a very recent characterization [10] of Jacobians that are isomorphic to a product of elliptic curves. We only highlight one point of the easy implication, in the following remark (1), to clarify for ourselves the distinction between isomorphism of polarized varieties, and non-.

**Theorem 1.** A genus-2 Riemann surface has a split Jacobian if and only if it has a canonical homology basis for which the period matrix \( Z \) satisfies:

\[
Z = \tau \begin{bmatrix} na & nb \\ nb & d \end{bmatrix}
\]

with \( \tau \) in the upper-half plane, and \( a, b, d, n \) positive integers such that \( ad - nb^2 = 1 \).

**Remark 1.** The proof of Theorem 2.1 is easy in the “if” direction. To show that such an abelian surface is isomorphic to the product of the two elliptic curves with matrices \( \tau \) and \( n\tau \) respectively, Earle observes that the \( 2 \times 4 \) matrix \( (I, Z) \) (with \( Z \) as in Theorem 2.1 and \( I \) the identity) satisfies:

\[
(I, Z) = (I, \tau V) \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}
\]

where

\[
V := \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}, \quad T := \tau \begin{bmatrix} a & b \\ nb & d \end{bmatrix}.
\]

In other words, there is a complex basis of \( \mathbb{C}^2 \) that’s a combination of the given basis and reduces the torus to the product of two elliptic curves. We know that this isomorphism cannot preserve the principal polarization,
because a Jacobian is not isomorphic to the product of two elliptic curves as a polarized variety. This comes down to the fact:

\[ J = \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} \neq J, \quad \text{where} \quad J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \]

**Remark 2.** Not all matrices of this form are period matrices of genus-2 curves; as Earle states, a real symmetric positive-definite \(2 \times 2\) matrix \(P\) is such that \(\tau P\) is a period matrix for some \(\tau\) if and only if there is no matrix \(A\) in \(SL(2, \mathbb{Z})\) such that \(AP^tA\) is diagonal. Earle also gives equivalent conditions for \(P\) to have this property. To exhibit one, it suffices to check that \(P := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\) has eigenvectors that are not defined over \(\mathbb{Q}\).

**Remark 3.** Not all the matrices of this form that are period matrices come from different curves; in two further theorems Earle gives criteria, if the matrix \(P\) is of a certain type, that tell whether the entries transform in such a way that the matrices come from the same curve via a different choice of homology basis.

Notably, [47] gives a method to tell whether a ppas is isomorphic or isogenous to a product of elliptic curves, that amounts to checking whether linear or quadratic equations on the entries of its \(2 \times 2\) period matrix have rational solutions. For Jacobians that are isomorphic to products, [46] gives an algorithm which in principle determines the curve (an algebraic equation), and [20] gives an algorithm to go the opposite way, namely to obtain the \(2 \times 2\) period matrix, given the \(j\)-invariants of the two elliptic curves.

In dimension 3, no such knowledge is available. Classically, Weierstrass asked Kovalevski to take up the problem of reduction and in her thesis she found a condition for degree-2 elliptic subcovers [28]. Her condition is analytic, and a condition of similar kind can be found in Riemann’s unpublished papers [45]-IV.B. Translated into algebra, her criterion is the following: a smooth plane quartic (the canonical image of a curve of genus 3) admits a degree-2 elliptic subcover if and only if four of its bitangents are concurrent [2]-VIII.76. More modern criteria are given in [21], where the group of rational points is the focus of investigation, and algebraic equations are provided for curves of genus 3 that have Jacobians isomorphic to the product of three quite special elliptic curves.

### 2.3 Examples and nonexamples.

For which covers of genus \(g\) does \(\text{Jac} X\) split further? In genus 3, a generic dimension count gives 4 for the variety of elliptic covers, and at most 3 for a variety isogenous to the product of three elliptic curves, so the generic cover will split into an elliptic curve and an abelian surface, which must be a Jacobian (since the property of splitting is meant up to isogeny); I am not aware of a general method that provides explicitly the curve of genus 2. Classes of examples were constructed in the nineteenth century, while explicitly integrating Hamiltonian systems, such as that which describes the motion of a solid body in a liquid (e.g., [26], and [27], which constructs what nowadays is called a Lax pair with an elliptic parameter as noted in [9]); these systems have been recently reprised in the context of algebraically completely integrable Hamiltonian systems.

- A nonexample of elliptic cover with Jacobian splittable into a product of elliptic curves. As proved in [1] (but not for the first time!) the curve \(Y\) of genus 2 with automorphism group of order 10 is not a Jacobian (since the property of splitting is meant up to isogeny); I am not aware of a general method that provides explicitly the curve of genus 2. Classes of examples were constructed in the nineteenth century, while explicitly integrating Hamiltonian systems, such as that which describes the motion of a solid body in a liquid (e.g., [26], and [27], which constructs what nowadays is called a Lax pair with an elliptic parameter as noted in [9]); these systems have been recently reprised in the context of algebraically completely integrable Hamiltonian systems.

- As for examples, it was known classically that the Klein curve (whose automorphism groups have maximum order 168) has Jacobian isomorphic to the product of three elliptic curves; it was proved again explicitly and algebraically in [42]. In [11], we observe that the Halphen curve of genus three,

\[ w^3 = \left( z^2 + \frac{25}{4}g_3 \right) \left( z^2 - \frac{135}{4}g_3 \right), \]
which has its origin in spectral theory and can be taken as the starting point of a Boussinesq hierarchy of PDEs, has Jacobian isogenous (thus, by [30]-[31], isomorphic) to the cube of the elliptic curve with \( j = 0, \psi^2 = 4\psi^3 + (40g_3)^2 \). It seems to happen frequently that curves coming from spectral problems have split jacobians, but we don’t have a precise statement at this time.

**Situation (III).** A naïve dimension count says that in \( M_g \) the subvariety of tangential covers has dimension \( g \). These assertions follow from the classification given by Treibich and Verdier [54]. The set of curves that are tangential covers is a subvariety because, given an elliptic curve with its origin \((E,q)\), all possible tangential covers make up an open subset of a linear system on an algebraic surface (whose complement in the closure consists of reducible curves). The system has dimension \( n + 1 \) where \( n \) is the degree of the cover, and the generic curve of the system has genus \( n \); but there is a 1-dimensional group action whose orbits consist of isomorphic curves. In this construction, one can vary the elliptic curve as well as the point on it which is taken to be the origin, so the dimension of the moduli space of covers goes up by one and down by one (this is the reason for the \( n + 2 \) in [57]-II.10.2). This is a naïve count because different elliptic curves could give rise to the same covers; for example, the situation in genus 2 is as follows: while there are zero, one or infinitely many elliptic subgroups of a ppas, there can only be a finite number of any given degree [23]. On the other hand, Krichever [29] showed that there is a completely integrable system whose integral manifolds are Jacobians of curves of genus \( g \), parametrizing elliptic KP solutions (and that each elliptic KP solution arises in this way). Again, unless there is a collapse in dimension, the action variables provide a \( g \)-dimensional variety of tangential covers. On the other hand, a parameter count à la Riemann says that the subvariety \( \mathcal{M}_{g}^{\text{eff}} \) of curves that cover an elliptic curve has dimension \( 2g - 2 \). Indeed, for covers of any degree, the Riemann-Hurwitz formula gives a ramification index of \( 2g - 2 \); for generic covers, the branching divisor will consist of \( 2g - 2 \) points. In analogy to a parameter count over \( \mathbb{P}^1 \), we must subtract the dimension of the possible maps from a given curve to the elliptic curve, but there can only be finite number of these, since upon fixing the origin, they correspond again to subgroups of the Jacobian. We refer the interested reader to the much more complete, and entirely rigorous, treatment of covers (of arbitrary genera) given in [32]-[33]. The statement that we are using is:

Let \( M_g \) be the moduli scheme (over \( \mathbb{Z} \)) for curves of genus \( g \), \( g \geq 2 \), and let \( M_g(g',n) \) be the set of geometric points of \( M_g \) such that for every \( P \in M_g(g',n) \) and curve \( \Gamma_P \) corresponding to \( P \), there exists a morphism \( f : \Gamma_P \to \Gamma' \) of degree \( n \) onto a curve \( \Gamma' \) of genus \( g' \). In characteristic 0, \( M_g(g',n) \) is equidimensional and \( \dim M_g(g',n) = 2g - 2 - (2n - 3) \cdot (g' - 1) \), for \( g > g' = 1, n \geq 2 \) and for \( g > g' > 1, (g - 1)/(g' - 1) \geq n \geq 2 \). In the remaining cases \( M_g(g',n) \) is empty or the whole of \( M_g \).

A comparison of dimensions, if nothing else, shows that when \( g > 2 \) an elliptic cover in general is not tangential; but it is difficult to construct explicit examples, more so in terms of the algebraic equation of the curve as opposed to the period matrix; I am providing the following two observations which are easy but I have not found in the literature.

- **A minimal elliptic subcover of degree 4.** I don’t know of an algebraic criterion to see whether a degree-4 elliptic subcover of a genus-2 curve is minimal; but the example of the curve defined on a suitable surface by the following “tangential polynomial” [53],

\[
T^4 + 3(e_i - 2\psi)T^2 + 4\psi/T - 3 \prod_{j \neq i}(\psi - e_j)
\]

(with \( i = 1, 2, 3, e_i \) finite branchpoints of the elliptic curve in Weierstrass form) must have this property by general facts that detect tangential covers embedded in a suitable surface [54].

- **A nonexample.** A split-Jacobian which does not give rise to elliptic solitons. Min-Ho Lee [35] calculates the expression of elliptic solutions for the KdV equation associated to the genus-4 curve: \( y^2 = x(x^4 - \alpha^4)(x^4 - \beta^4) \), by splitting its period matrix (the large automorphism group of the curve allows for calculation of the period matrix, cf. [34]-Ch.11, and this example can be seen to be isogenous to the product of four elliptic curves). But note that this is not an elliptic soliton. Indeed, [52]-Prop. 2.1 provides a necessary and sufficient condition for a cover \( X \), viewed as a field extension of the field
of elliptic functions, to be tangential: the local parameter on $X$ corresponding to the KdV flow $\frac{d^2z}{dt^2}$ would have to map to a local parameter $z$ on the elliptic subcover, in such a way that the algebraic equation for $X$ is a polynomial whose coefficients are elliptic functions regular outside $z = 0$ and whose poles in $z$ have a bounded behavior.

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**References**


