A. L. KOSTKO, A. V. TSIGANOV
Department of Mathematical and Computational Physics
Institute of Physics
St. Petersburg University
198 904, St.Petersburg, Russia
E-mail: tsiganov@mph.phys.spbu.ru

NONCANONICAL TRANSFORMATIONS OF THE SPHERICAL TOP

Received February 4, 2003
DOI: 10.1070/RD2003v008n02ABEH000233

We discuss noncanonical transformations connecting different integrable systems on the symplectic leaves of the Poisson manifolds. The special class of transformations, which consists of the symplectic mappings of symplectic leaves and of the parallel transports induced by diffeomorphisms in the base of symplectic foliation, is considered. As an example, we list integrable systems associated with the spherical top. The corresponding additional integrals of motion are second, third and six order polynomials in momenta.

1. Introduction

Integrable Hamiltonian system on a symplectic manifold consists of a Hamiltonian that generates the dynamics together with a Lagrangian fibration (distribution or polarization) such that the flow lines generated by the Hamiltonian are parallel to the fibers.

For the role of Lagrangian foliations in geometric quantization on symplectic manifolds see e.g. [5], [6]. It is known that connection (or parallel transport) preserving Lagrangian fibration is symplectic connection, i.e. torsionless connection which preserve a given symplectic form. In the theory of integrable systems, it means that such connection gives rise to canonical transformations, which map a given integrable system to equivalent ones.

The Poisson manifolds are a natural generalization of symplectic manifolds. From the geometric quantization theory we know that there are connections which parallelize Lagrangian fibrations but do not preserve the Poisson structure (see references in [5]). The corresponding noncanonical transformation maps a given integrable system to another integrable system.

Thus on the Poisson manifolds we could to apply noncanonical transformation to the known integrable models in order to get new integrable systems. In this paper, we consider some examples of such transformations.

Note, in the deformation quantization theory the main problem is construction of the canonical connection which parallelize two transversal Lagrangian distributions and preserve symplectic or Poisson structure [5], [6]. So, we have not any description of connections, which do not preserve the Poisson structure and parallelize a given Lagrangian distribution.

2. Integrable systems on Poisson manifolds

On the symplectic manifold $\mathcal{M}$, $\dim \mathcal{M} = 2n$ integrable system $(H, \mathcal{F}_I)$ consists of Hamilton function $H$ and Lagrangian fibration

$$\mathcal{F}_I : \mathcal{M} \to B^n.$$
The fibers $F(\alpha)$ of $\mathcal{F}_i$ are Lagrangian submanifolds, i.e. each leaf has $\dim F = n$ and $\Omega(X, Y) = 0,$ for all $X, Y$ tangent to $F$. Usually, the fibers $F(\alpha)$ are level surfaces $I_k = \alpha_k$ of function $I_k$ called integrals of motion [8].

The regular transversally constant Poisson manifold $\mathcal{M}$ turns out to be foliated in symplectic leaves $\mathcal{O}_a$ of codimension $k$

$$\mathcal{F}_a : \mathcal{M} \to \mathcal{A}^k$$

Below we shall consider such Poisson manifolds for which the minimal Poisson submanifolds $\mathcal{O}_a$ leaves are level surfaces of the Casimir functions $C_i, i = 1, \ldots, k$ on $\mathcal{M}$, which are independent and constant $C_i = a_i$ on each symplectic leaf.

Integrable system on the Poisson manifold is an integrable system on the symplectic leaves of the Poisson manifold. It consists of the Hamilton function $H$, which may be function on the whole manifold, and Lagrangian fibrations $\mathcal{F}_i$ on some symplectic leaves. Roughly speaking it is composition of the symplectic and lagrangian fibrations $\mathcal{F}_i \circ \mathcal{F}_s$. It allows us to transform integrable system using compositions of symplectic transformations $\rho_a : \mathcal{O}_a \to \mathcal{O}_a$ and parallel transports $T_{ab} : \mathcal{O}_a \to \mathcal{O}_b$ induced by the diffeomorphisms $a_i \to b_i$ in the base of symplectic fibration $\mathcal{F}_s$.

Let $(H, \mathcal{F}_i)$ be integrable system on some symplectic leaf $\mathcal{O}_a$ of the Poisson manifold $\mathcal{M}$. Any symplectic mapping $\rho_a$ on $\mathcal{O}_a$ parallelize Lagrangian fibration and sends a given integrable system to equivalent ones

$$\rho_a : (H, \mathcal{F}_i) \to (\tilde{H}, \tilde{\mathcal{F}}_i).$$

If there are parallel transport $T_{ab} : \mathcal{O}_a \to \mathcal{O}_b$, which sends resulting Lagrangian fibration on $\mathcal{O}_a$ to Lagrangian fibration on $\mathcal{O}_b$ depending on $n$ parameters

$$\tilde{\mathcal{F}}_i = T_{ab} \tilde{\mathcal{F}} = T_{ab} \rho_a F_i,$$

one gets new integrable system with the Hamiltonian $\tilde{H} = T_{ab} \rho_a H$ on new symplectic leaf $\mathcal{O}_b$.

The new system $(\tilde{H}, \tilde{\mathcal{F}}_i)$ will be different from initial one $(H, \mathcal{F}_i)$, if the parallel transport $T_{ab}$ does not permutable with the symplectic transformation $\rho_a$, i.e. there no exists symplectic transformation $\rho_b$ on $\mathcal{O}_b$ such that

$$T_{ab} \rho_a = \rho_b T_{ab}. \quad (2.2)$$

In this case, transformation $T_{ab} \rho_a$ preserves a given Lagrangian fibration and does not keep Poisson structure. In other words, one gets noncanonical transformation which relates different integrable systems on different symplectic manifolds.

The parallel translations $T_{ab}$ are completely determined by the Poisson manifold $\mathcal{M}$. So, the conditions (2.1) and (2.2) are the equations for the symplectic transformations $\rho_a$ and Lagrangian fibrations $\mathcal{F}_i$, which may be useful to construction of new integrable systems.

The aim of this paper is to consider some solutions of these equations (2.1) and (2.2) for a Poisson manifold.

3. Integrable systems on algebra $e(3)$

In this section we discuss some integrable systems in a rigid body dynamics which can be described by the following equations

$$J = J \times \frac{\partial H}{\partial J} + x \times \frac{\partial H}{\partial x}, \quad \dot{x} = x \times \frac{\partial H}{\partial J} \quad (3.1)$$

They are so-called the Euler-Poisson equations for the motion of a rigid body in a constant gravity or the Kirchhoff equations for the motion of a rigid body in an ideal fluid [3].
There are six dynamical variables: three components of the angular momentum $J = (J_1, J_2, J_3)$ and three components of the gravity vector $x = (x_1, x_2, x_3)$ (Poisson vector), everything with respect to a moving orthogonal frame attached to the body.

It is well known [1], that equations (3.1) are Hamiltonian equations

$$J_t = \{J, H\}, \quad x_t = \{x, H\}$$

with respect to the Lie-Poisson brackets of the Euclidean $e(3)$ algebra

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \quad \{x_i, x_j\} = 0,$$

where $\varepsilon_{ijk}$ is the totally skew-symmetric tensor. These brackets have two Casimir functions

$$C_1 = (x, x) = |x|^2, \quad C_2 = (J, x),$$

which is a four-dimensional symplectic manifold. This Poisson structure is the reduction of the standard symplectic structure on $T^*SO(3)$ with respect to rotations around the vertical axis [1]. The symplectic leaves $O_{a\ell}$ are topologically equivalent to cotangent bundle $T^*S^2$ of the sphere $S^2$ defined by $x_1^2 + x_2^2 + x_3^2 = a^2$.

Thus, the equations (3.1) are Hamiltonian equations on the reduced four-dimensional phase space.

Let us consider some Poisson maps which preserve the Poisson structure and change the form of the Hamilton function only.

**Example.** If $f(J_3)$ and $g(J_3)$ are arbitrary functions, satisfying the condition $f^2 + g^2 = \text{const}$, the map

$$\rho : \quad x \to \tilde{x} = Ux, \quad J \to \tilde{J} = \frac{U J}{\sqrt{f^2 + g^2}},$$

where

$$U = \begin{pmatrix} f & g & 0 \\ -g & f & 0 \\ 0 & 0 & \sqrt{f^2 + g^2} \end{pmatrix}$$

is nonlinear Poisson mapping of the Poisson manifold $e(3)$. This mapping sends the Hamilton function for the Kowalevski top

$$H = J_1^2 + J_2^2 + 2J_3^2 + x_1$$

to a new function

$$\tilde{H} = J_1^2 + J_2^2 + 2J_3^2 + f(J_3) x_1 + g(J_3) x_2,$$

one of the possible forms of which may be as the following

$$\tilde{H} = J_1^2 + J_2^2 + 2J_3^2 + \sin(J_3) x_1 + \cos(J_3) x_2.$$

The Hamilton functions (3.7), (3.8) and (3.9) have a different form up to canonical transformation (3.6).

Let us discuss the construction of noncanonical transformations. Among generic symplectic leaves $O_{a\ell}$ (3.5) we single out two subsets of symplectic leaves

$$A_\ell = \{O_{a\ell}, \quad a = 0\} \quad \text{and} \quad B_\ell = \{O_{a\ell}, \quad \ell = 0\}$$

for the Kowalevski top.
Recall, for the Euler-Poisson equation the Casimir function $C_1 = a^2$ (3.4) has a purely geometrical origin such as $a$ is a radius of the Poisson sphere acting as a reduced configuration space. The second Casimir function $C_2 = \ell$ (3.4) is a principal angular momentum in horizontal plane, which is so-called area integral. For the Kirchhoff equations the Casimir functions $C_1$, $C_2$ are so-called impulse force integral and impulse momentum integral, respectively.

So, on the leaves $\mathcal{A}_\ell$ integrable systems describe dynamics on the Poisson sphere with zero radius and, therefore, the corresponding solutions of the Euler-Poisson and Kirchhoff equations have not any physical meaning. On the other hand, on the second family of symplectic leaves $B_a$ integrable Hamiltonians describe dynamics on a Poisson sphere with some finite radius $a$ and, therefore, solutions of the Euler-Poisson and Kirchhoff equations represent a certain physical interest.

Below we consider noncanonical transformations which relate non-physical orbits $\mathcal{A}_\ell$ and physical orbits $B_a$. Let $I_j$ are integrals of motion for some integrable system on $\mathcal{A}_\ell$. Acting the symplectic mapping $\rho_f$ (3.13) on $I_j$ one gets another functions $\tilde{I}_j$, which Poisson commute on symplectic leaves $\mathcal{A}_\ell$

$$\{\tilde{I}_j, \tilde{I}_k\}_{\mathcal{A}_\ell} = \{\rho_f I_j, \rho_f I_k\}_{\mathcal{A}_\ell} = \rho_f \{I_j, I_k\}_{\mathcal{A}_\ell} = 0.$$  

Then we can apply to these integrals the parallel transport $T_{a\ell}: \mathcal{A}_\ell \to B_a$ of symplectic leaves induced by diffeomorphisms $(0, \ell) \to (a, 0)$ at the base of symplectic fibration. If the resulting functions $\tilde{I}_j$ are in the involution on symplectic leaves $B_a$,

$$\{\tilde{I}_j, \tilde{I}_k\}_{B_a} = \{T_{ab} \rho_f I_j, T_{ab} \rho_f I_k\}_{B_a} = 0. \tag{3.11}$$

one gets some integrable system on $B_a$ related with original integrable system on $\mathcal{A}_\ell$.

The principal conditions (3.11) are equations on symplectic mapping $\rho_f$ (3.13) and initial integrals of motion $I_j$. We have not general solution of the proposed equations (3.11).

In order to construct some partial solutions of these equations (3.11) we begin with construction of a special family of symplectic mappings on $\mathcal{A}_\ell$. In [11] similar transformations were considered as noncanonical transformations of Lie-Poisson brackets connected with monopole.

**Proposition 1.** For any solution of the following PDE

$$f(x) + \sum_{j=1}^{3} x_j \frac{\partial f(x)}{\partial x_j} = 0, \tag{3.12}$$

the mapping

$$\rho_f : \quad x \to x, \quad J \to J + x f(x), \tag{3.13}$$

is the symplectic mapping, which preserves symplectic structure of each leaf $\mathcal{A}_\ell$ (3.10) by $C_1 = a^2 = 0$.

Proof.

Substituting new coordinates $\tilde{J}_k = J_k + x_k f(x)$ into the Poisson brackets (3.3) it is easy to verify that $\rho_f$ is the Poisson mapping on $\mathcal{A}_\ell$ iff $f$ is solution of (3.12). On the other hand by $C_1 = 0$ the mapping $\rho_f$ preserve the value of the second Casimir function

$$\tilde{C}_2 = (x, \tilde{J}) = (x, J) + |x|^2 f(x) = C_2 + C_1 f(x) = C_2.$$

So, mapping (3.13) sends each leaf $\mathcal{A}_\ell$ to itself and, therefore, it is a symplectic mapping of every leaf $\mathcal{A}_\ell$.

Equation (3.12) is well known homogeneity equation, which has a general solution

$$f(x) = \sum_{i,j} \varphi_{ij}(x_i x_j^{-1}) \tag{3.14}$$

where $\varphi_{ij}(z)$ is an arbitrary function.
Moreover, we know a special integral of equation (3.12)
\[ f(x) = (x_1x_2x_3)^{-1/3}, \]  
which cannot be received from the general solution [14].

Now we could substitute mapping \( \rho f \) into the system (3.11) and solve the resulting equations with respect to unknown integrals \( I_j \). In the search of such solutions we use relationship between the Kowalevski exponents and Liouville integrability [7], [17]. Instead of the search integrals of motion we are looking for the Hamilton functions for which the Kowalevski exponents are invariant with respect to our noncanonical transformation.

In this paper we apply the corresponding noncanonical transformations to the spherical top and list the resulting integrable systems on \( B_a \).

4. Deformations of the spherical top

The original Hamilton function for the spherical top is equal to
\[ H = J_1^2 + J_2^2 + J_3^2. \]
Symplectic mapping \( \rho_f \) (3.13) sends this function to another function
\[ \tilde{H}_f = \rho_f H = J_1^2 + J_2^2 + J_3^2 + 2\ell f(x) + a^2 f^2(x). \]  
(4.1)
According to Proposition 1, if \( a = 0 \) and \( f(x) \) satisfies equation (3.12), this Hamiltonian \( \tilde{H}_f \) is integrable on the symplectic leaves \( A_t \).

The corresponding Hamilton function \( T_{ab} \tilde{H}_f \) on \( B_a \) is equal to
\[ \tilde{H} = J_1^2 + J_2^2 + J_3^2 + |x|^2 f^2(x) \]  
(4.2)
Recall, that on \( B_a \) we have to put \( \ell = (x, J) = 0 \) and \( |x|^2 = a^2 \).

**Remark.** Let two three-dimensional vectors \((y, J)\) with the Lie-Poisson bracket
\[ \{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{x_i, x_j\} = \kappa \varepsilon_{ijk} x_k, \]  
(4.3)
are coordinates on the Lie algebra \( so(4) \) [3]. According to [15], the mapping determined by the equation
\[ \frac{y}{|y|} = \frac{x}{|x|}, \quad \text{iff} \quad (x, J) = (y, J) = 0, \]  
(4.4)
is the Poisson map, which relates the Poisson manifolds \( so(4) \) and \( e(3) \). This map preserve the form of Hamilton function \( H \) (4.2) and the Hamilton function on \( so(4) \) has the same form
\[ \tilde{H}_\kappa = J_1^2 + J_2^2 + J_3^2 + |y|^2 f^2(y). \]  
(4.5)
The Poisson map is canonical transformation which preserves integrability and, therefore, all the obtained results will be correct for both integrable Hamiltonians (4.2) and (4.5) on the Poisson manifolds \( e(3) \) and \( so(4) \), respectively.

4.1. The Kowalevski analysis

The Kowalevski method and the Painlevé \( \alpha \) method share a common underlying principle, the integrability of differential equation is somehow related to the single-valuedness of the solution and one of the crucial ingredients of these analyses is the powers at which arbitrary coefficients appear in the local series around the singularities, that is the Kowalevski exponents. The Kowalevski exponents have an interesting relationship with the Liouville integrability which is discussed in [7], [16], [17].
The Kowalevski exponents are used to build local series solutions around movable singularities of (3.1), which are the Laurent series or the Puiseux series when all the positive Kowalevski exponents are integers or rational, respectively. In order to define the Kowalevski exponents we have to study the linearized equations of motion (3.1) around the scale invariant solutions

\[ x_i = x_i^{(0)} t^{\nu_i}, \quad J_i = J_i^{(0)} t^{\mu_i}, \quad i = 1, 2, 3. \]  

(4.6)

The corresponding Jacobian matrix evaluated on the solutions (4.6) is so-called Kowalevski matrix. The eigenvalues of this matrix are the Fuchs indices at a regular singularity \( t = 0 \) and they are so-called Kowalevski exponents of the nonlinear system (3.1).

The Kowalevski exponents for the original spherical top are \( \{-1, -1, -1, 1, 1, 1\} \). For the new Hamiltonian \( \tilde{H} \) (4.2) \( \mu_i = 1 \) and \( \nu_i = -1 \) and the Kowalevski matrix has a block structure

\[
\mathcal{K} = \text{diag}(1, 1, 1, -1, -1, -1) + \left( \begin{array}{c}
\mathcal{X} \\
\mathcal{J}
\end{array} \right),
\]

where

\[
\mathcal{X} = 2 \left( \begin{array}{ccc}
0 & x_3^{(0)} & -x_2^{(0)} \\
-x_3^{(0)} & 0 & x_1^{(0)} \\
x_2^{(0)} & -x_1^{(0)} & 0
\end{array} \right), \quad \mathcal{J} = -2 \left( \begin{array}{ccc}
0 & J_3^{(0)} & -J_2^{(0)} \\
-J_3^{(0)} & 0 & J_1^{(0)} \\
J_2^{(0)} & -J_1^{(0)} & 0
\end{array} \right),
\]

and

\[
\mathcal{M}_{ij} = -a^2 \partial \frac{\partial f^2(x_1, x_2, x_3)}{\partial x_i} \bigg|_{x_i = x_i^{(0)}}.
\]

Recall that here \( x_i^{(0)} \) and \( J_i^{(0)} \) are general solutions of the linearized equations of motion (3.1).

We can not find these solutions and the corresponding Kowalevski exponents for the general function \( f \) (3.14). Nevertheless, we can suppose that the following hypothesis is true

**Assumption.** For all the systems with the Hamiltonian \( \tilde{H} \) (4.2) the Kowalevski exponents are \( \{-2, -1, 0, 0, 1, 2\} \).

We can prove this result for some concrete general solutions (3.14) and for the special integral (3.15) of equation (3.12) only.

As an example, let us consider the Hamiltonian \( \tilde{H} \) (4.2) with

\[
f = \frac{1}{x_j} \varphi \left( \frac{x_i}{x_j} \right), \quad \varphi \left( \frac{x_i}{x_j} \right) = \left( \frac{x_i}{x_j} \right)^n, \quad n \in \mathbb{N}. \]

(4.7)

For these systems matrices \( \mathcal{M}_n \) are defined by the following recurrence relation:

\[
\mathcal{M}_n = \mathcal{M}_{n-1} \left( \frac{x_i}{x_j} \right)^2 + \left( \mathcal{M}_1 x_i x_j^3 - \mathcal{M}_0 x_i^3 x_j + 4(n - 1)x_k \mathcal{V} \right) \left( \frac{x_i^{2n-3}}{x_j^{2n+1}} \right).
\]

Here \( k \) is the complement to \( i, j \) in the set \( \{1, 2, 3\} \) and

\[
\mathcal{V} = \left( \begin{array}{ccc}
-a^2 & \frac{a^2 x_i}{x_j} & 0 \\
-\frac{a^2 x_i}{x_j} & a^2 & 0 \\
\frac{a^2 (x_i^2 + x_j^2)}{x_i x_k} & -\frac{a^2 (x_i^2 + x_j^2)}{x_j x_k} & 0
\end{array} \right).
\]

Substituting the corresponding solutions of the linearized equations of motion (3.1) into this matrix we can prove that for all the systems (4.7) the Kowalevski exponents are \( \{-2, -1, 0, 0, 1, 2\} \).
4.2. Construction of the integrals of motion

In this section our main aim is a practical construction of the integrals of motion \( \hat{I}_j = T_{ab} \rho_f I_j \) on the symplectic leaves \( B_n \) starting with known integrals of motion \( I_j \) on the symplectic leaves \( A_\ell \).

Below we use the following two projectors

\[ P = \frac{1}{2}(\rho_f + \rho_f^{-1}) \quad \text{and} \quad P^* = \frac{1}{2}(\rho_f^* + \rho_f^{*-1}), \]  

(4.8)

where \( \rho_f^{-1} : J \to J - x f(x) \) and \( \rho_f^* : J \to J + ix f(x). \)

It is easy to verify that they act trivially in the subspace of linear functions

\[ PJ_k = P^* J_k = J_k \]  

and in the subspace of quadratic functions

\[ \frac{1}{2}(P + P^*)J_k^2 = J_k^2, \quad \frac{1}{2}PP^* J_k^2 = J_k^2. \]

We shall use these maps for the construction of the integrals of motion, which are linear, quadratic or cubic polynomials in momenta \( J \), according to the rule

\[ \hat{I}_1 = PI_1, \quad \hat{I}_2 = P^* I_2. \]

(4.9)

For instance, for the deformations (4.2) of the spherical top one gets

\[ \hat{I}_1 = \hat{H} = PH \]

In the next sections using some concrete functions \( f \) in the mappings \( \rho_f \) we add to this Hamilton function another integrals of motion \( \hat{I}_2 \), which are second, third and sixth order polynomials in momenta.

4.3. Quadratic integrals

At first, let us review general solution \( f(x) \) (3.14) of the equation (3.12).

**Proposition 2.** Dynamical system with the Hamiltonian \( \hat{H} \) (4.2) is integrable by \( \ell = 0 \), if

\[ f(x) = \frac{1}{x_i} \varphi \left( \frac{x_i}{x_j} \right) \quad \text{and} \quad \hat{H} = J_1^2 + J_2^2 + J_3^2 + \frac{a^2}{x_i^2} \varphi^2 \left( \frac{x_i}{x_j} \right). \]

(4.10)

Additional integral of motion is given by

\[ \hat{I}_2 = J_k^2 + \frac{x_i^2 + x_j^2}{x_i^2} \varphi^2 \left( \frac{x_i}{x_j} \right) \]

(4.11)

where \( k \) is the supplement to \( i, j \) in the set \( \{1, 2, 3\} \) and \( \varphi \) is an arbitrary function.

The proof consists of direct calculation of Poisson brackets \( \{\hat{H}, \hat{I}_2\} = 0 \). Additional integral \( \hat{I}_2 \) is constructed from the known quadratic integral for the original spherical top \( I_2 = J_k^2 - a^2 f^2(x) \simeq J_k^2 \) on \( A_\ell \) according to the rule (4.9).

For the general function \( f \) (3.14) in (4.2) the Kowalevski exponents and the formal solutions around movable singularities have a similar form. However we can not construct the corresponding additional integral of motion in the generic case.

The additional integral we can find for the deformations of the generalized Lagrange top

\[ H = J_1^2 + J_2^2 + J_3^2 + g(x_3), \quad I_2 = J_3^2, \]

where \( g(x_3) \) is an arbitrary function.
Proposition 3. Dynamical system with the Hamilton function

$$\hat{H} = PH = J_1^2 + J_2^2 + J_3^2 + g(x_3) + \frac{a_2}{x_1} \varphi^2 \left( \frac{x_1}{x_2} \right),$$

(4.12)

is integrable by $\ell = 0$ and additional integral of motion is given by

$$\hat{I}_2 = J_3^2 + \frac{x_1^2 + x_2^2}{x_1^2} \varphi^2 \left( \frac{x_1}{x_2} \right)$$

(4.13)

where $\varphi(x_1/x_2)$ is an arbitrary function.

In this case the Kowalevski exponents became irrational such as for the original Lagrange top by $g(x_3) = x_3$ the Kowalevski exponents are $\{0, \frac{1}{2}, 1, 2, 2\}$. So, the irrationality of the Kowalevski exponents can not be related in general to Liouville nonintegrability.

4.3.1. The Lax matrices

For all these integrable systems both integrals are quadratic polynomials in $J$ and, therefore, these systems belong to the Stackel family of integrable systems. Thus, $2 \times 2$ Lax representation may be constructed according to a general scheme [13].

As an example, we show the Lax matrix for the Rosochatius system. Let us consider composition $\rho_{f_1} \rho_{f_2} \rho_{f_3}$ of the symplectic transformations $\rho_{f_i}$ (3.13) with trivial functions $\varphi_i = c_i$, i.e. under $f_i = c_i/x_i$.

Applying this composition of mappings to the Hamiltonian of the Neumann system [10]

$$H = J_1^2 + J_2^2 + J_3^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2, \quad a_i \in \mathbb{R}$$

one gets the Hamilton function for the so-called Rosochatius system

$$\hat{H} = J_1^2 + J_2^2 + J_3^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + |x|^2 \left( \frac{c_1^2}{x_1^2} + \frac{c_2^2}{x_2^2} + \frac{c_3^2}{x_3^2} \right), \quad c_i \in \mathbb{R}. \quad (4.14)$$

The integrability of this system was established by Rosochatius in 1877 using the separation of variables method [12].

Our aim is to rewrite the flow with the Hamiltonian $\hat{H}$ (4.14) as the following matrix differential equation

$$\frac{d}{dt} L(\lambda) = [L(\lambda), A]. \quad (4.15)$$

Let $p_i, x_i$ be canonical variables on manifold $\mathbb{R}^6$ with a Poisson bracket $\{p_i, x_j\} = \delta_{ij}$. The mapping defined by $J_i = \epsilon_{ijk} x_j p_k$ is a Poisson map $\mathbb{R}^6 \rightarrow B_a$. The Lax matrix for the Rosochatius system has a more compact form in canonical variables $(x, p)$

$$L(\lambda) = \begin{pmatrix} -\partial_t b/2 & b \\ -\partial_t b/2 \partial_t b/2 & 2(x, p) b \end{pmatrix} - \frac{1}{4|x|^2} \begin{pmatrix} 2(x, p) b & 0 \\ |x|^2 u + 2(x, p) \partial_t b + (p, p) b & -2(x, p) b \end{pmatrix}, \quad (4.16)$$

where

$$\partial_t = \frac{1}{4|x|^2} \{[J]^2, \cdot \}, \quad u = 1 + \sum_{i=1}^{3} \frac{x_i^2}{x_i^2 (\lambda - a_i)}.$$

This Lax matrix $L(\lambda)$ is completely defined by a single matrix entry

$$b = \frac{x_1^2}{\lambda - a_1} + \frac{x_2^2}{\lambda - a_2} + \frac{x_3^2}{\lambda - a_3} = \prod_{i=1}^{3} \frac{(\lambda - Q_i)}{\prod (\lambda - a_i)}, \quad (4.17)$$
which determines transformation to the separated variables $Q_i$ [13]. Using the Hamilton function (4.14) in (4.23) one gets

$$L(\lambda) = \sum_{i=1}^{3} \left( -\frac{1}{2} \frac{x_i p_i}{\lambda - a_i} + \frac{x_i^2}{\lambda - a_i} \right) - \frac{1}{4} \left( 1 + \sum_{i=1}^{3} \frac{c_i^2}{x_i^2(\lambda - a_i)} \right)$$

(4.18)

The Lax matrices for integrable systems (4.10) and (4.12) have the similar form and may be constructed in framework of the general scheme [13].

**Remark.** Using the sum to some number $n$ in this formulae one gets the Lax matrix for the Rosochatius system on the $n - 1$-dimensional sphere $S^{n-1}$.

### 4.4. Degenerated systems

Among a given family of integrable system we single out two degenerate or superintegrable systems. The first one with the Hamiltonian

$$\hat{H} = J_1^2 + J_2^2 + J_3^2 + a^2 \frac{c_1 x_2 x_3 + c_2 (x_3^2 + x_2^2)}{(x_3^2 - x_2^2)^2}$$

has two functionally independent additional integrals of motion

$$\hat{K}_1 = J_1^2 + \frac{x_2 x_3 (c_1 (x_3^2 + x_2^2) + 4c_2 x_2 x_3)}{(x_3^2 - x_2^2)^2},$$

$$\hat{K}_2 = J_2 J_3 + \frac{x_1^2 (c_1 (x_3^2 + x_2^2) + 4c_2 x_2 x_3)}{4(x_3^2 - x_2^2)^2}.$$ (4.20)

The second one represents a particular case of the Rosochatius system (4.14)

$$\hat{H} = J_1^2 + J_2^2 + J_3^2 + |x|^2 \left( \frac{c_1^2}{x_1^2} + \frac{c_2^2}{x_2^2} + \frac{c_3^2}{x_3^2} \right).$$ (4.21)

Three additional integrals of motion are quadratic polynomials in $J$

$$\hat{K}_m = J_m^2 + c_2 x_i^2 x_j + c_2 x_j^2 x_i,$$

where $m$ is the complement to $i, j$ in the set $\{1, 2, 3\}$.

#### 4.4.1. The Lax matrices

In [9] Mozer studied the Lax representation for the Rosochatius system on the $(n - 1)$-dimensional sphere $S^{n-1}$. In this section we construct another Lax representation for a special degenerate case of the Rosochatius system (4.21).

Let $\{x, p\}_{i=1}^{n}$ are canonical variables and the Hamilton function is equal to

$$\hat{H} = \sum_{i,j=1}^{n} J_{ij}^2 + \sum_{k=1}^{n} \frac{c_k^2}{x_k^2}, \quad J_{ij} = p_i x_j - x_i p_j.$$ (4.22)

This system coincides with our superintegrable system (4.21) by $n = 3$. 
The known Lax matrix [9] for the flow generated by the Hamiltonian (4.22)

\[ L = \mathcal{J} + \mathcal{V} = p \wedge x + x \odot \frac{c}{x} + \frac{c}{x} \odot x \]  

(4.23)
gives rise to integrals of motion \( \hat{I}_k = \text{trace } L^k \), which are quadratic polynomials in momenta \( p \).

Here matrix \( \mathcal{J} = p \wedge x \) is the antisymmetric matrix constituted from the components of angular momentum \( J \)

\[ J_{ij} = p_ix_j - x_ip_j \]  

(4.24)
and matrix \( \mathcal{V} = x \odot c/x + c/x \odot x \) is the symmetric matrix with the following entries

\[ V_{ij} = \frac{c_j x_i}{x_j} + \frac{c_i x_j}{x_i} = \frac{c_j x_i^2 + c_i x_j^2}{x_ix_j}, \quad V_{ii} = 0 \]  

(4.25)

Using some counterpart of the Fourier transformation [14] we can construct another Lax matrix which gives rise to integrals of motion of higher degree.

**Proposition 4.** The flow with the Hamiltonian \( \hat{H} \) (4.22) is equivalent to the Lax equation (4.15), where first Lax matrix is equal to

\[ \hat{L} = \hat{\mathcal{J}} + \hat{\mathcal{V}}, \quad \hat{L}_{ij} = \text{sign}(j-i) L_{ij}. \]  

(4.26)

Here \( \hat{\mathcal{J}} \) is symmetric matrix and \( \hat{\mathcal{V}} \) is antisymmetric matrix with the same matrix elements (4.24) and (4.25), respectively

\[ \hat{J}_{ij} = \text{sign}(j-i) J_{ij}, \quad \hat{V}_{ij} = \text{sign}(j-i) V_{ij}. \]

The second Lax matrix \( \hat{A} \) is the diagonal matrix

\[ \hat{A} = 2|x|^2 \text{diag} \left( \frac{c_1}{x_1^2}, \frac{c_2}{x_2^2}, \ldots, \frac{c_n}{x_n^2} \right) \]

The integrals of motion \( \hat{I}_k = \text{trace } \hat{L}^k \) are the \( k \)-degree polynomials in momenta \( p \).

In the next section we consider another example of integrable system with the similar Lax matrix.

4.5. Integrals of higher degree

Let us consider a special integral (3.15) of equation (3.12).

**Proposition 5.**

Dynamical system with Hamilton function \( \hat{H} \) (4.2) is integrable by \( \ell = 0 \) if

\[ f(x) = b(x_1x_2x_3)^{-1/3}, \quad \hat{H} = J_1^2 + J_2^2 + J_3^2 + \frac{|x|^2b^2}{(x_1x_2x_3)^{2/3}}, \quad b \in \mathbb{C}. \]  

(4.27)

Additional integral of motion is given by

\[ \hat{K}_b = P^*(J_1J_2J_3) = J_1J_2J_3 + \frac{b^2}{(x_1x_2x_3)^{2/3}} \left( x_2x_3J_1 + x_1x_3J_2 + x_1x_2J_3 \right). \]  

(4.28)

Additional integral of motion \( \hat{K}_b \) is constructed by the rule (4.9) starting with the following integral of motion \( I_2 = J_1J_2J_3 \) for the original spherical top.
The corresponding $3 \times 3$ Lax matrices

$$
\tilde{L} = \tilde{\mathcal{F}} + \tilde{\mathcal{V}}, \quad \tilde{A} = \frac{2|x|^2 f(x)}{3} \begin{pmatrix}
0 & x_3^{-1} & x_2^{-1} \\
x_3 & 0 & x_1^{-1} \\
x_2^{-1} & x_1^{-1} & 0
\end{pmatrix}.
$$

were constructed in [14]. As for the Rosochatius system (4.26), here $\tilde{\mathcal{F}}$ is the symmetric matrix with entries (4.24) and $\tilde{\mathcal{V}}$ is the antisymmetric matrix with the following entries

$$
\tilde{V}_{ij} = \text{sign}(j - i) b \frac{(x_1x_2x_3)^{2/3}}{x_ix_j}.
$$

In contrast with the Rosochatius system we can not generalize this Lax matrix to $(n - 1)$-dimensional sphere $S^{n-1}$.

In order to construct generalization of this integrable system we add to the Hamilton function (4.27) the first superintegrable potential (4.19). Namely, let us apply the composition $\rho_b \rho_c$ of symplectic transformation $\rho$ (3.13) to the Hamilton function of the spherical top, where

$$
f_b = b (x_1x_2x_3)^{-1/3}, \quad f_c = \frac{c \sqrt{x_3^2 + x_2^2}}{x_3^2 - x_2^2}
$$

are the particular integral $f_b$ and the general integral $f_c$ of the equation (3.12), which corresponds to the degenerate motion along closed trajectories (4.19) under the following conditions $c_1 = 0, c_2 = c^2$.

**Proposition 6.** Dynamical system with Hamilton function

$$
\tilde{H} = J_1^2 + J_2^2 + J_3^2 + \frac{|x|^2 b^2}{(x_1x_2x_3)^{2/3}} + \frac{|x|^2 c^2 (x_2^2 + x_3^2)}{(x_2^2 - x_3^2)^2}
$$

(4.29)

is integrable by $\ell = 0$. The second integral of motion is the six order polynomial in momentum

$$
\tilde{I}_2 = \tilde{K}_b^2 + \tilde{K}_1 \tilde{K}_2 + \tilde{K}_d,
$$

where $\tilde{K}_{1,2}$ and $\tilde{K}_b$ are given by the formulas (4.20) and (4.28) under $c_1 = 0, c_2 = c^2$ and additional item $\tilde{K}_d$ is equal to

$$
\tilde{K}_d = -J_1^2 J_2^2 J_3^2 + \frac{(2f_b f_c x_1x_2x_3)^2}{(x_2^2 + x_3^2)} \left[ f_c^2 \left( x_1^2 - \frac{2x_2^2 x_3^2}{x_2^2 + x_3^2} \right) + f_b^2 (x_1^2 - x_3^2)(x_1^2 - x_2^2) + \right.$$

$$
\left. + \frac{2(x_2^2 + x_3^2) J_2 J_3 - x_2 x_3 J_1^2 - x_1 J_1 (x_2 J_3 + x_3 J_2)}{2x_2x_3} - \frac{2x_2x_3 J_2 J_3}{x_1^2} \right].
$$

(4.30)

The integrability of a given system was proved in [4]. As above, after change of variables $x \to y$ the same Hamilton function $\tilde{H}$ (4.29) describes an integrable system on the Poisson manifold $so(4)$ by $(y, J) = 0$.

5. Conclusions

Integrable system on the Poisson manifold is the Lagrangian fibration on the symplectic submanifolds. Composition of the symplectic mapping of symplectic leaf with the parallel transport is noncanonical transformation in generic. If such transformation parallelizes a given Lagrangian fibration one gets noncanonical transformation which connects two different integrable systems.

In this paper, we study some integrable systems associated with the spherical top by a special family of noncanonical transformations. In all the considered cases these transformations preserve the Kovalevski exponents and a form of the local series solutions around a singularity. However we found additional integrals of motion in some special cases only. The corresponding additional integrals of motion are second, third and six order polynomials in momenta.
References


