MULTIPLE HAMILTONIAN STRUCTURES
FOR TODA SYSTEMS OF TYPE A–B–C

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Results on the finite nonperiodic $A_n$ Toda lattice are extended to the Bogoyavlensky–Toda systems of type $B_n$ and $C_n$. The investigated areas include master symmetries, recursion operators, higher Poisson brackets and invariants. A conjecture which relates the degrees of higher Poisson brackets and the exponents of the corresponding Lie group is verified for these systems.

1. Introduction

The purpose of this paper is to show that the hierarchy of Poisson brackets for the Bogoyavlensky–Toda systems of type $A_n$, $B_n$ and $C_n$ is connected with fundamental invariants of the corresponding Lie group, namely the exponents of the Lie group. This observation was predicted a long time ago (1986) by Hermann Flaschka, but at that time not too many people believed it would be correct. The Toda lattice is a system of particles on the line where each particle interacts with its neighbour with an exponential force. The original Toda system with an infinite number of particles was considered by Toda [34] in 1967. The integrability of the system is due to Flaschka [12], Henon [17] and Manakov [23], all in 1974. The explicit solution of the finite lattice is due to Moser [25] in 1975. We restrict our attention to the finite, non-periodic version of the Toda lattice. In the process we will investigate some other related topics, e.g., bi-Hamiltonian structure, recursion operators, symmetries, master symmetries, Lax formulations, Poisson and symplectic Geometry. This area of integrable systems has been studied extensively also for infinite dimensional systems such as the KdV, Burgers, Kadomtsev–Petviashvili, Benjamin–Ono equations and many more.

The natural setting for Hamiltonian systems is on symplectic manifolds. These are Poisson manifolds which Poisson structure is locally isomorphic to the standard one on $\mathbb{R}^{2N}$. A Poisson structure on a manifold $M$ may be defined as a contravariant tensor field (bivector) $\pi$ for which the Poisson bracket on $C^\infty(M)$, $\{f, g\} = \langle \pi, df \wedge dg \rangle$ satisfies the Jacobi identity. When $\pi$ has a full rank, the dimension of $M$ is even and the Poisson structure is symplectic. Darboux’s theorem provides coordinates which make the structure locally isomorphic to the standard symplectic bracket on $\mathbb{R}^{2N}$.

To define a Hamiltonian system we consider $\mathbb{R}^{2N}$ with coordinates $(q_1, \ldots, q_N, p_1, \ldots, p_N)$, and the standard symplectic bracket
\[
\{f, g\} = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).
\]

Let $H : \mathbb{R}^{2N} \to \mathbb{R}$ be a smooth function. Hamilton equations are the differential equations
\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.
\]
Using the symplectic bracket (1.1), Hamilton equations take the simple form
\[ \dot{F} = \{F, H\}. \]  
(1.3)
The condition \( \dot{F} = 0 \) is equivalent to the condition \( \{F, H\} = 0 \). Such a function is called a constant of motion (or the first integral).

Smooth functions \( f_1, \ldots, f_r \) on a manifold \( M \) are called independent, if \( df_1 \wedge \cdots \wedge df_r \neq 0 \), except possibly on a submanifold of smaller dimension. The functions \( f_1, \ldots, f_r \) are said to be mutually in involution if \( \{f_i, f_j\} = 0 \), \( \forall i, j \). A Hamiltonian system is called integrable if there exists a family \( f_1, \ldots, f_r \) of independent \( C^\infty \) functions mutually in involution.

The equations for the Toda systems in consideration will be written in the form
\[ \dot{L}(t) = [B(t), L(t)]. \]  
(1.4)
The pair of matrices \( L, B \) is known as a Lax pair. In the case of the finite nonperiodic Toda lattice \( L \) is a symmetric tridiagonal matrix and \( B \) is the projection onto the skew-symmetric part in the decomposition of \( L \) into skew-symmetric plus lower triangular. In the case of \( B_n \) and \( C_n \) Toda systems the matrix \( L \) will lie in the corresponding Lie algebra and \( B \) will again be obtained from \( L \) by some projection associated with a decomposition of the Lie algebra. The decomposition plays an important role in the solution of the equations by factorization.

In the case of Toda lattice the Lax equation is obtained by the use of a transformation due to H. Flaschka [12] which changes the original \((p, q)\) variables to new reduced variables \((a, b)\). The symplectic bracket in the variables \((p, q)\) transforms to a degenerate Poisson bracket in the variables \((a, b)\). This linear bracket is an example of a Lie–Poisson bracket. The functions \( H_n = \frac{1}{n} \text{tr} L^n \) are in involution. A Lie algebraic interpretation of this bracket can be found in [19]. We denote this bracket by \( \pi_1 \). A quadratic Toda bracket, which we call \( \pi_2 \) appeared in the paper of Adler [1]. It is a Poisson bracket in which the Hamiltonian vector field generated by \( H_1 \) is the same as the Hamiltonian vector field generated by \( H_2 \) with respect to the \( \pi_1 \) bracket. This is an example of a bi-Hamiltonian system, an idea introduced by Magri [22]. A cubic bracket was found by Kupershmidt [20] via the infinite Toda lattice. We found the explicit formulas for both the quadratic and cubic brackets in some lecture notes by H. Flaschka. The Lenard relations (3.13) are also in these notes.

In [3] we used master symmetries to generate nonlinear Poisson brackets for the Toda lattice. In essence, we have an example of a system which is not only bi-Hamiltonian but it can actually be given \( N \) different Hamiltonian formulations with \( N \) as large as we please. The first three Poisson brackets are precisely the linear, quadratic and cubic brackets we mentioned above, but one can use the master symmetries to produce an infinite hierarchy of brackets. If a system is bi-Hamiltonian and one of the brackets is symplectic, one can find a recursion operator by inverting the symplectic tensor. The recursion operator is then applied to the initial symplectic bracket to produce an infinite sequence. However, in the case of Toda lattice (in Flaschka variables \((a, b)\)) both operators are non-invertible and therefore this method fails. The absence of a recursion operator for the finite Toda lattice is also mentioned in Morosi and Tondo [24] where a Ninjenhuis tensor for the infinite Toda lattice is calculated. Recursion operators were introduced by Olver [30]. Master symmetries were first introduced by Fokas and Fuchssteiner in [15] in connection with the Benjamin–Ono Equation. Then W. Oevel and B. Fuchssteiner [28] found master symmetries for the Kadomtsev–Petviashvili equation. The general theory of master symmetries is discussed in Fuchssteiner [16]. In the case of Toda equations the master symmetries map invariant functions to other invariant functions. Hamiltonian vector fields are also preserved. New Poisson brackets are generated by using Lie derivatives in the direction of these vector fields and they satisfy interesting deformation relations. Another approach, which explains these relations is adopted in Das and Okubo [6], and Femandes [11]. In principle, their method is general and may work for other finite dimensional systems as well. The procedure is the following: One defines a second Poisson bracket in the space of canonical variables \((q_1, \ldots, q_N, p_1, \ldots, p_N)\). This gives
rise to a recursion operator. The presence of a conformal symmetry as defined in Oevel [29] allows one, by using the recursion operator, to generate an infinite sequence of master symmetries. These, in turn, project to the space of the new variables \((a, b)\) to produce a sequence of master symmetries in the reduced space. This approach was also used in [27] by da Costa and Marle in the case of the Relativistic Toda lattice.

The Toda lattice has been generalized in several directions: Bogoyavlensky [2] and Kostant [19] generalized the system to the tridiagonal coadjoint orbit of the Borel subgroup of an arbitrary simple Lie group. Therefore, for each simple Lie group there is a corresponding mechanical system of Toda type.

Another generalization is due to Deift, Li, Nanda and Tomé [7] who showed that the system remains integrable when \(L\) is replaced by a full (generic) symmetric \(n \times n\) matrix.

Another variation is the full Kostant-Toda lattice which was studied by S. Singer, N. Ercolani and H. Flaschka [8], [14], [33]. In this case, the matrix \(L\) in the Lax equation is the sum of a lower triangular plus a regular nilpotent matrix.

We also mention the relativistic Toda lattice which was introduced by Ruijsenaars [32]. The non-relativistic Toda lattice can be thought as a limiting case of this system.

There is also extensive research on a system which is closely related to the Toda lattice. It is known as the Volterra system or KM-system. The Volterra model and its Poisson structure is treated in detail in [9]. The phase space consists of variables \(u_n\), with \(u_n > 0\). The equations of motion are:

\[
\frac{du_n}{dt} = (u_{n+1} - u_n)u_n,
\]

where \(a_0 = a_{N+1} = 0\). These equations appeared first in [35], to describe population evolution in a hierarchical system of competing individuals. This system is interesting because it can be considered as a discrete analogue of the Korteweg-de Vries equation. It also appears in the discretization of conformal field theory; the Poisson bracket for this system can be thought of as a lattice generalization of the Virasoro algebra [10]. The variables \(u_n\) are intermediate step in constructions angle-action variables for the Liouville model on the lattice. This system was solved by Kac and Van Moerbeke [18] following the method of Flaschka [13] which uses a discrete version of inverse scattering (see also ref. [26]). The Volterra model and the Toda model are closely related: There exists a transformation due to Hénon which maps one system to the other. The mapping is given by

\[
a_i = -\frac{1}{2}\sqrt{u_{2i-1}u_{2i-2}}, \quad b_i = \frac{1}{2}(u_{2i-1} + u_{2i-2}).
\]

The equations satisfied by the new variables \(a_i, b_i\) are given by:

\[
\dot{a}_i = a_i(b_{i+1} - b_i), \quad \dot{b}_i = 2(a_i^2 - a_{i-1}^2).
\]

These are precisely the Toda equations (3.4) which we derive later on. The multiple Hamiltonian structures, Lax pairs and master symmetries for this system were derived in [4].

In Section 2 we present the necessary background on Poisson manifolds, bi-Hamiltonian systems and master symmetries. We also define the exponents of a simple Lie group.

Section 3 is a review of the classical finite nonperiodic Toda lattice. This system was investigated in [12], [13], [17], [25], [34]. We define the quadratic and cubic Toda brackets and show that they satisfy certain Lenard-type relations. We briefly describe the construction of master symmetries and the new Poisson brackets as in [3], [5]. We also describe the method of R. Fernandes [11].

In Section 4 we define some integrable systems associated with simple Lie groups. They have been considered by Bogoyavlensky [2], Kostant [19] and Olshanetsky and Perelomov [31]. We present in detail the systems of type \(B_n\) and \(C_n\). We show that they are bi-Hamiltonian and we also construct a recursion operator and master symmetries. The main result is the following: For the Toda systems of type \(A_n, B_n\) and \(C_n\), the degrees of the higher Poisson brackets coincide with the exponents of the corresponding Lie group.
2. Background

Poisson Manifolds

The theory of Poisson manifolds can be traced back to Sophus Lie but it was popularized recently by Lichnerowicz and Weinstein [21], [36].

Let $M$ be a $C^\infty$ manifold, $N = C^\infty(M)$ the algebra of $C^\infty$ real valued functions on $M$. A contravariant, antisymmetric tensor of order $p$ will be called a $p$-tensor for short.

A Poisson structure on $M$ is a bilinear form, called the Poisson bracket $\{ , \} : N \times N \to N$ such that

i) \[ \{ f, g \} = -\{ g, f \}, \] (2.1)

ii) \[ \{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0, \] (2.2)

iii) \[ \{ f, gh \} = \{ f, g \} h + \{ f, h \} g. \] (2.3)

Properties i) and ii) define a Lie algebra structure on $N$. ii) is called the Jacobi identity and iii) is the analogue of Leibniz rule from calculus. A Poisson manifold is a manifold $M$ together with a Poisson bracket $\{ , \}$.

To a Poisson bracket one can associate a 2-tensor $\pi$ such that

\[ \{ f, g \} = \langle \pi, df \wedge dg \rangle. \] (2.4)

Jacobi’s identity is equivalent to the condition $[\pi, \pi] = 0$ where $[ , ]$ is the Schouten bracket. Therefore, one could define a Poisson manifold by specifying a pair $(M, \pi)$ where $M$ is a manifold and $\pi$ a 2-tensor satisfying $[\pi, \pi] = 0$. In local coordinates $(x_1, x_2, \ldots, x_n)$, $\pi$ is given by

\[ \pi = \sum_{i,j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \] (2.5)

and

\[ \{ f, g \} = \langle \pi, df \wedge dg \rangle = \sum_{i,j} \pi_{ij} \frac{\partial f}{\partial x_i} \wedge \frac{\partial g}{\partial x_j}. \] (2.6)

In particular $\{ x_i, x_j \} = \pi_{ij}(x)$. Knowledge of the Poisson matrix $(\pi_{ij})$ is sufficient to define the bracket of arbitrary functions.

A function $F: M_1 \to M_2$ between two Poisson manifolds is called a Poisson mapping if

\[ \{ f \circ F, g \circ F \}_1 = \{ f, g \}_2 \circ F \] (2.7)

for all $f, g \in C^\infty(M_2)$. In terms of tensors, $F_\ast \pi_1 = \pi_2$. Two Poisson manifolds are called isomorphic, if there exists a diffeomorphism between them which is a Poisson mapping.

The Poisson bracket allows one to associate a vector field to each element $f \in N$. Property iii) implies that $\{ f, . \}$ is a derivation of $N$. Hence, for each $f \in N$ there exists a well defined vector field $\chi_f$ defined by the formula

\[ \chi_f(g) = \{ f, g \}. \] (2.8)

It is called the Hamiltonian vector field generated by $f$. 

Hamiltonian vector fields are *infinitesimal automorphisms* of the Poisson structure. These are vector fields $X$ satisfying $L_X \pi = 0$. In the case of Hamiltonian vector fields we have

$$L_X \pi = 0. \tag{2.9}$$

The Hamiltonian vector fields form a Lie algebra and in fact

$$[X_f, X_g] = X_{(f \circ g)}.$$ \tag{2.10}

So, the map $f \mapsto X_f$ is a Lie algebra homomorphism.

The Poisson structure defines a bundle map $\pi^*: T^*M \to TM$ such that

$$\pi^*(df) = \chi_f. \tag{2.11}$$

The functions in the center of $N$ are called *Casimirs*. It is the set of functions $f$ so that $\{f, g\} = 0$ for all $g \in N$. These are functions which are constant along the orbits of Hamiltonian vector fields. The differentials of these functions are in the kernel of $\pi^*$.

**Examples**

The most basic examples of Poisson brackets are the symplectic and Lie–Poisson brackets.

i) **Symplectic manifolds**: A *symplectic manifold* is a pair $(M^{2n}, \omega)$ where $M^{2n}$ is an even dimensional manifold and $\omega$ is a closed, non-degenerate two-form. The associated isomorphism

$$\mu: TM \to T^*M$$ \tag{2.12}

extends naturally to a tensor bundle isomorphism still denoted by $\mu$. Let $\lambda = \mu^{-1}$, $f \in N$ and let $\chi_f = \lambda(df)$ be the corresponding Hamiltonian vector field. The symplectic bracket is given by

$$\{f, g\} = \omega(\chi_f, X_g). \tag{2.13}$$

In the case of $\mathbb{R}^{2n}$, there are coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, so that

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i \tag{2.14}$$

and the Poisson bracket is the standard one (1.1).

ii) **Lie Poisson**: Let $M = \mathcal{G}^*$ where $\mathcal{G}$ is a Lie algebra. For $a \in \mathcal{G}$ define the function $\Phi_a$ on $\mathcal{G}^*$ by

$$\Phi_a(\mu) = \langle a, \mu \rangle \tag{2.15}$$

where $\mu \in \mathcal{G}$ and $\langle \cdot, \cdot \rangle$ is the pairing between $\mathcal{G}$ and $\mathcal{G}^*$. Define a bracket on $\mathcal{G}^*$ by

$$\{\Phi_a, \Phi_b\} = \Phi_{[a, b]}. \tag{2.16}$$

This bracket is easily extended to arbitrary $C^\infty$ functions on $\mathcal{G}^*$. The bracket of linear functions is linear and every linear bracket is of this form, i.e., it is associated with a Lie algebra.

Let $(M, \pi_1)$, $(M, \pi_2)$ be two Poisson structures on $M$. The two brackets are called compatible if $\pi_1 + \pi_2$ is Poisson. Suppose $\pi_1$ is Poisson and $\pi_2 = L_X \pi_1$ for some vector field $X$. Then it follows easily that $\pi_1$ is compatible with $\pi_2$. If $\pi_1$ is symplectic, we call the Poisson pair non-degenerate. If we assume a non-degenerate pair we make the following definition: The *recursion operator* associated with a non-degenerate pair is the $(1,1)$-tensor $\mathcal{R}$ defined by

$$\mathcal{R} = \pi_2 \wedge \pi_1^{-1}. \tag{2.17}$$

A *bi-Hamiltonian system* is defined by specifying two Hamiltonian functions $H_1$, $H_2$ satisfying:

$$X = \pi_1 \nabla H_2 = \pi_2 \nabla H_1. \tag{2.18}$$

We have the following result due to Magri [22].
Theorem 1. Suppose we have a bi-Hamiltonian system on a manifold $M$, the first cohomology group of which is trivial. Then there exists a hierarchy of mutually commuting functions $H_1, H_2, \ldots$, all in involution with respect to both brackets. They generate mutually commuting bi-Hamiltonian flows $\chi_i, i = 1, 2, \ldots$, satisfying the Lenard recursion relations
\[
\chi_{i+j} = \pi_i \nabla H_j,
\]
where $\pi_{i+1} = R^i \pi_1$ are the higher order Poisson tensors.

Master Symmetries

We recall the definition and basic properties of master symmetries following Fuchssteiner [16].

Consider a differential equation on a manifold $M$, defined by a vector field $\chi$. We are mostly interested in the case where $\chi$ is a Hamiltonian vector field. A vector field $Z$ is a symmetry of the equation if
\[
[Z, \chi] = 0.
\]
If $Z$ is time dependent, then a more general condition is
\[
\frac{\partial Z}{\partial t} + [Z, \chi] = 0.
\]
A vector field $Z$ will be called a master symmetry if
\[
[[Z, \chi], \chi] = 0,
\]
but
\[
[Z, \chi] \neq 0.
\]
Suppose that we have a bi-Hamiltonian system defined by the Poisson tensors $J_0, J_1$ and the Hamiltonians $h_0, h_1$. Assume that $J_0$ is symplectic. We define the recursion operator $R = J_1 J_0^{-1}$, the higher flows
\[
\chi_i = R^{i-1} \chi_1,
\]
and the higher order Poisson tensors
\[
J_i = R^i J_0.
\]
Master symmetries preserve constants of motion, Hamiltonian vector fields and generate hierarchies of Poisson structures. For a non-degenerate bi-Hamiltonian system, master symmetries can be generated using a method due to W. Oevel [29].

Theorem 2. Suppose that $Z_0$ is a conformal symmetry for both $J_0, J_1$ and $h_0$, i.e. for some scalars $\lambda, \mu, \nu$ we have
\[
L_{Z_0} J_0 = \lambda J_0, \quad L_{Z_0} J_1 = \mu J_1, \quad L_{Z_0} h_0 = \nu h_0.
\]
Then the vector fields
\[
Z_i = R^i Z_0
\]
are master symmetries and we have
\[
(a) \quad [Z_i, \chi_j] = (\mu + \nu + (j - 1)(\mu - \lambda)) \chi_{i+j},
\]
\[
(b) \quad [Z_i, Z_j] = (\mu - \lambda)(j - i) Z_{i+j},
\]
\[
(c) \quad L_{Z_i} J_j = (\mu + (j - i - 1)(\mu - \lambda)) J_{i+j}.
\]
Exponents

Let us recall the definition of exponents for a semisimple group $G$. Let $G$ be a connected complex simple Lie group $G$. We form the de Rham cohomology groups $H^i(G, \mathbb{C})$ and the corresponding Poincaré polynomial of $G$:

$$p_G(t) = \prod_i (1 + t^{2e_i+1}).$$

(2.27)

The positive integers $\{e_1, e_2, \ldots, e_l\}$ are called the exponents of $G$. One can also extract the exponents from the root space decomposition of $G$. The connection with the invariant polynomials is the following: Let $H_1, H_2, \ldots, H_l$ be algebraically independent homogeneous polynomials of degrees $n_1, n_2, \ldots, n_l$. Then $n_i = e_i + 1$. The exponents of a simple Lie algebra are given in the following list:

- $A_{n-1}$: $1, 2, 3, \ldots, n - 1$,
- $B_n, C_n$: $1, 3, 5, \ldots, 2n - 1$,
- $D_n$: $1, 3, 5, \ldots, 2n - 3, n - 1$,
- $G_2$: $1, 5$,
- $F_4$: $1, 5, 7, 11$,
- $E_6$: $1, 4, 5, 7, 8, 11$,
- $E_7$: $1, 5, 7, 9, 11, 13, 17$,
- $E_8$: $1, 7, 11, 13, 17, 19, 23, 29$.

3. Finite, non-periodic $A_n$ Toda lattice

The Toda lattice is a completely integrable classical mechanical system consisting of $n$ particles on the line and subject to a system of springs which behave exponentially.

$$H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \sum_{j=1}^{N-1} e^{q_j-q_{j+1}}.$$ 

(3.1)

The function $q_j(t)$ is the position of the $j$th particle and $p_j(t)$ is the corresponding momentum.

There are two other types of chains:

1. The infinite chain, where $-\infty \leq j \leq \infty$ with conditions at infinity.
2. The periodic chain where $q_1 = q_{N+1}$.

We restrict our attention to the finite, non-periodic version of the Toda lattice.
Hamilton equations (1.2) become
\[ \ddot{q}_j = p_j, \quad \dot{p}_j = e^{q_{j-1} - q_j} - e^{q_j - q_{j+1}}. \] (3.2)
This system is integrable. One can find a set of independent functions \{H_1, \ldots, H_N\} which are constants of motion for Hamilton equations. To determine the constants of motion, one uses Flaschka’s transformation:
\[ a_i = \frac{1}{2} e^{q_{i-1} - q_i}, \quad b_i = -\frac{1}{2} p_i. \] (3.3)
Then
\[ \dot{a}_i = a_i (b_{i+1} - b_i), \quad \dot{b}_i = 2 (a_i^2 - a_{i-1}^2). \] (3.4)
These equations can be written as a Lax pair \( \dot{L} = [B, L] \), where \( L \) is the symmetric Jacobi matrix
\[ L = \begin{pmatrix}
    b_1 & a_1 & 0 & \cdots & \cdots & 0 \\
    a_1 & b_2 & a_2 & \cdots & \cdots & \\
    0 & a_2 & b_3 & \cdots & \cdots & \\
    \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & a_{N-1} & b_N \\
\end{pmatrix}, \] (3.5)
and
\[ B = \begin{pmatrix}
    0 & a_1 & 0 & \cdots & \cdots & 0 \\
    -a_1 & 0 & a_2 & \cdots & \cdots & \\
    0 & -a_2 & 0 & \cdots & \cdots & \\
    \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & a_{N-1} & 0 \\
\end{pmatrix}. \] (3.6)
As we showed in the introduction, a Lax equation implies that the traces of powers of \( L \) are constants of motion. This implies that the eigenvalues of \( L \) do not evolve with time. Therefore, such systems will be integrable.

Remark. The Lax equation
\[ \dot{L}(t) = [B(t), L(t)], \quad L(0) = L_0 \]
can be solved by factorization. First we perform a Gram-Schmidt factorization \( e^{tL_0} = k(t) b(t) \), where \( k(t) \) is orthogonal and \( b(t) \) upper triangular. The solution is given by
\[ L(t) = k(t)^{-1} L_0 k(t). \]
The form of the solution shows again that the eigenvalues of \( L \) are independent of \( t \).

Consider \( \mathbb{R}^{2N} \) with coordinates \((q_1, \ldots, q_N, p_1, \ldots, p_N)\), the standard symplectic bracket \( \{f, g\}_s \), and the mapping \( F: \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2N-1} \) defined by
\[ F: (q_1, \ldots, q_N, p_1, \ldots, p_N) \rightarrow (a_1, \ldots, a_{N-1}, b_1, \ldots, b_N). \] (3.7)
Define a bracket on \( \mathbb{R}^{2N-1} \) by
\[ \{f, g\} = \{f \circ F, g \circ F\}_s. \]
The result is a bracket determined by

\[ \{a_i, b_i\} = -a_i, \quad \{a_i, b_{i+1}\} = a_i. \]  \hfill (3.8)

All other brackets are zero. \( H_1 = b_1 + b_2 + \cdots + b_N \) is the only Casimir. The Hamiltonian turns out to be \( H_2 = \frac{1}{2} \text{tr} \, L^2 \) and the functions \( H_j \) are in involution. We denote this Poisson tensor by \( \pi_1 \).

The quadratic Toda bracket appears in conjunction with isospectral deformations of Jacobi matrices. First, let \( \lambda \) be an eigenvalue of \( L \) with normalized eigenvector \( v \). Standard perturbation theory shows that

\[ \nabla \lambda = (2v_1 v_2, \ldots, 2v_{N-1} v_N, v_1^2, \ldots, v_N^2)^T = U^\lambda, \]  \hfill (3.9)

where \( \nabla \lambda \) denotes \( (\frac{\partial \lambda}{\partial a_1}, \ldots, \frac{\partial \lambda}{\partial b_N}) \). Some manipulations show that \( U^\lambda \) satisfies

\[ \pi_2 U^\lambda = \lambda \pi_1 U^\lambda, \]  \hfill (3.10)

where \( \pi_2 \) and \( \pi_1 \) are skew-symmetric matrices. It turns out that \( \pi_1 \) is the matrix of coefficients of the Poisson tensor (3.8), and \( \pi_2 \), whose coefficients are quadratic functions of the \( a_i \)'s and \( b_i \)'s, can be used to define a new Poisson tensor. This bracket appeared in a paper of Adler [1] in 1979. The defining relations for the new bracket \( \pi_2 \) are:

\[ \{a_i, a_{i+1}\} = \frac{1}{2} a_i a_{i+1}, \quad \{a_i, b_i\} = -a_i b_i, \quad \{a_i, b_{i+1}\} = a_i b_{i+1}, \quad \{b_i, b_{i+1}\} = 2 a_i^2; \]  \hfill (3.11)

all other brackets are zero. This bracket has \( \det L \) as Casimir and \( H_1 = \text{tr} \, L \) is the Hamiltonian. The eigenvalues of \( L \) are still in involution and

\[ \pi_2 \nabla \lambda_j = \lambda_j \pi_1 \nabla \lambda_j, \quad \forall j. \]  \hfill (3.12)

It follows easily that

\[ \pi_2 \nabla H_t = \pi_1 \nabla H_{t+1}. \]  \hfill (3.13)

These relations are similar to the Lenard relations for the KdV equation. They show that the Toda lattice is a bi-Hamiltonian system.

Since it is impossible to find a recursion operator for the non-periodic Toda lattice we use a different method to generate invariants. The idea is to define master symmetries, and use Lie derivatives to generate higher invariants.

We describe the construction following [3], [5]. We denote the master symmetries by \( X_n \). These vector fields generate an infinite sequence of contravariant 2-tensors \( \pi_n \), for \( n \geq 1 \). We summarize the properties of \( X_n \) and \( \pi_n \):

**Theorem 3.**

1. \( \pi_n \) are all Poisson.
2. The functions \( H_n = \frac{1}{n!} \text{tr} \, L^n \) are in involution with respect to all of the \( \pi_n \).
3. \( X_n(H_m) = (n + m)H_{n+m} \).
4. \( L_{X_n} \pi_m = (m - n - 2)\pi_{n+m} \).
5. \( \pi_n \nabla H_t = \pi_{n-1} \nabla H_{t+1} \), where \( \pi_n \) denotes the Poisson matrix of the tensor \( \pi_n \).

To define the vector fields \( X_n \) we consider expressions of the form

\[ \hat{L} = [B, L] + L^n. \]  \hfill (3.14)

This equation is similar to a Lax equation, but in this case the eigenvalues satisfy \( \hat{\lambda} = \lambda^n \) instead of \( \lambda = 0 \).

As we mentioned earlier a recursion operator for the Toda lattice in Flaschka’s variables does not exist. However, there is another method of finding the master symmetries \( X_n \) due to Fernandes [11] which we describe briefly:
The first step is to define a second Poisson bracket on the space of canonical variables $(q_1, \ldots, q_N, p_1, \ldots, p_N)$. This bracket appears in Das and Okubo [6] and Fernandes [11]. We follow the notation from [11]. Let $J_0$ be the symplectic bracket and define $J_1$ as follows:

$$\{q_i, q_j\} = 1, \quad \{p_i, q_i\} = p_i, \quad \{p_i, p_{i+1}\} = e^{h_i - q_{i+1}}.$$  \hspace{1cm} (3.15)

Also define

$$h_0 = \sum_{i=1}^{N} p_i, \quad h_1 = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i=1}^{N-1} e^{h_i - q_{i+1}}.$$  \hspace{1cm} (3.16)

The recursion operator is defined by $\mathcal{R} = J_1 J_0^{-1}$. It follows easily that the vector field

$$Z_0 = \sum_{i=1}^{N} \frac{N+1-2i}{2} \frac{\partial}{\partial q_i} + \sum_{i=1}^{N} p_i \frac{\partial}{\partial p_i}$$  \hspace{1cm} (3.17)

is a conformal symmetry for $J_0$, $J_1$ and $h_0$ and therefore, Oevel’s theorem applies. The constants in Theorem 2 turn out to be $\lambda = -1$, $\mu = 0$ and $\nu = 1$. We end up with the following deformation relations:

$$L_{Z_i} J_j = (j - i - 1) J_{i+j}, \quad [Z_i, Z_j] = (j - i) Z_{i+j}, \quad [Z_i, \chi_j] = j \chi_{i+j}.$$  \hspace{1cm} (3.18)

Recall the Flaschka transformation (3.3) $F: \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2N-1}$ defined by

$$F: (q_1, \ldots, q_N, p_1, \ldots, p_N) \rightarrow (a_1, \ldots, a_{N-1}, b_1, \ldots, b_N).$$

The Poisson tensors $J_0$ and $J_1$ reduce to $\mathbb{R}^{2N-1}$. They reduce precisely to the tensors $\pi_1$ and $\pi_2$ (3.8), (3.11). The mapping $F$ is a Poisson mapping between $J_0$ and $\pi_1$. It is also a Poisson mapping between $J_1$ and $\pi_2$. The Hamiltonians $h_0$ and $h_1$ correspond to the reduced Hamiltonians $H_1$ and $H_2$ respectively. The recursion operator $\mathcal{R}$ cannot be reduced. Actually, it is easy to see that there exists no recursion operator in the reduced space. The kernels of the two Poisson structures $\pi_1$ and $\pi_2$ are different and, therefore, it is impossible to find an operator that maps one to the other.

The deformation relations (3.18) become precisely the deformation relations of Theorem 3. Of course, one has to replace $j$ by $j - 1$ in the formulas involving $J_k$ because of the difference in notation between [3] and [11].

4. Orthogonal Toda systems

Definition of the systems

In this section we consider mechanical systems which generalize the finite, nonperiodic Toda lattice. These systems correspond to Dynkin diagrams. It is well known that irreducible root systems classify simple Lie groups. So, in this generalization for each simple Lie algebra there exists a mechanical system of Toda type.

The generalization is obtained from the following simple observation: In terms of the natural basis $q_i$ of weights, the simple roots of $A_{n-1}$ are

$$q_1 - q_2, q_2 - q_3, \ldots, q_{n-1} - q_n.$$  \hspace{1cm} (4.1)

On the other hand, the potential for the Toda lattice is of the form

$$e^{q_1 - q_2} + e^{q_2 - q_3} + \cdots + e^{q_{n-1} - q_n}.$$  \hspace{1cm} (4.2)

We note that the angle between $q_{i-1} - q_i$ and $q_i - q_{i+1}$ is $\frac{2\pi}{3}$ and the lengths of $q_i - q_{i+1}$ are all equal. The Toda lattice corresponds to a Dynkin diagram of type $A_{n-1}$. 
More generally, we consider potentials of the form
\[ U = c_1 e^{f_1(q)} + \cdots + c_l e^{f_l(q)}, \]
where \( c_1, \ldots, c_l \) are constants, \( f_l(q) \) is linear and \( l \) is the rank of the simple Lie algebra. For each Dynkin diagram we construct a Hamiltonian system of Toda type. These systems are interesting not only because they are integrable, but also for their fundamental importance in the theory of semisimple Lie groups. For example Kostant in [19] shows that the integration of these systems and the theory of the finite dimensional representations of semisimple Lie groups are equivalent.

For reference, we give a complete list of the Hamiltonians for each simple Lie algebra.

For \( A_{n-1} \)
\[ H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n}, \]

For \( B_n \)
\[ H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_n}, \]

For \( C_n \)
\[ H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{2q_n}, \]

For \( D_n \)
\[ H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_{n-1}+q_n}, \]

For \( G_2 \)
\[ H = \frac{1}{2} \sum_{j=1}^{3} p_j^2 + e^{q_1-q_2} + e^{-2q_1+q_2+q_3}, \]

For \( E_6 \)
\[ H = \frac{1}{2} \sum_{j=1}^{8} p_j^2 + e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3} + e^{\frac{1}{2}(q_1-q_2+q_3-q_4)}, \]

For \( E_7 \)
\[ H = \frac{1}{2} \sum_{j=1}^{8} p_j^2 + \sum_{j=1}^{5} e^{q_j-q_{j+1}} + e^{-q_1+q_2} + e^{\frac{1}{2}(-q_1+q_2+\cdots+q_7-q_8)}, \]

For \( E_8 \)
\[ H = \frac{1}{2} \sum_{j=1}^{8} p_j^2 + \sum_{j=1}^{6} e^{q_j-q_{j+1}} + e^{-q_1+q_2} + e^{\frac{1}{2}(-q_1+q_2+\cdots+q_8)}\]

We should note that the Hamiltonians in the list are not unique. For example, the \( A_2 \) Hamiltonian is
\[ H = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + \frac{1}{2} p_3^2 + e^{q_1-q_2} + e^{q_2-q_3}. \]

An equivalent system is
\[ H(Q_1, P_1) = \frac{1}{2} P_1^2 + \frac{1}{2} P_2^2 + \frac{1}{2} P_3^2 + \sqrt{\frac{2}{3}}(\sqrt{3}Q_1 + Q_2) + e^{-2\sqrt{\frac{2}{3}} Q_2}. \]

The second Hamiltonian is obtained from the first by using the canonical transformation
\[
Q_1 = \frac{\sqrt{2}}{4}(q_1 + q_2 - 2q_3), \quad P_1 = \frac{2}{\sqrt{2}}(p_1 + p_2), \quad Q_2 = \frac{\sqrt{6}}{4}(q_2 - q_1), \quad P_2 = \frac{2}{\sqrt{6}}(p_2 - p_1).
\]
A recursion operator for $B_n$ Toda systems

In this section, we show that higher polynomial brackets exist also in the case of $B_n$ Toda systems. We will prove that these systems possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets in which the constants of motion are in involution.

The Hamiltonian for $B_n$ is

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \epsilon^{a-\epsilon_2} + \ldots + \epsilon^{a-n} + \epsilon^{a+n}. \quad (4.1)$$

We make a Flaschka-type transformation

$$a_i = \frac{1}{2} e^{2(a-a_{i+1})}, \quad a_n = \frac{1}{2} e^{2n}, \quad b_i = -\frac{1}{2} p_i. \quad (4.2)$$

Then

$$\dot{a}_i = a_i (b_{i+1} - b_i), \quad i = 1, \ldots, n, \quad \dot{b}_i = 2 (a_i^2 - a_{i-1}^2) \quad i = 1, \ldots, n, \quad (4.3)$$

with the convention that $a_0 = b_{n+1} = 0$. These equations can be written as a Lax pair $\dot{L} = [B, L]$, where $L$ is the symmetric matrix

$$\begin{pmatrix}
    b_1 & a_1 & \cdots & a_{n-1} \\
    a_1 & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & a_n \\
    a_{n-1} & b_n & a_n & 0 \\
    a_n & 0 & -a_n & \ddots \\
    \vdots & \ddots & \ddots & \ddots & a_n \\
    a_{n-1} & -a_n & -b_n & \ddots & \ddots \\
    -a_n & -b_n & \ddots & \ddots & a_n \\
    -a_1 & b_1 & \cdots & a_{n-1}
\end{pmatrix}, \quad (4.4)$$

and $B$ is the skew-symmetric part of $L$.

In the new variables $a_i$, $b_i$ the symplectic bracket $\pi_1$ is given by

$$\{a_i, b_i\} = -a_i, \quad \{a_i, b_{i+1}\} = a_i. \quad (4.5)$$

The invariant polynomials for $B_n$, which we denote by

$$H_2, H_4, \ldots, H_{2n}$$

are defined by $H_{2i} = \frac{1}{2i} \text{Tr} L^{2i}$. The degrees of the first $n$ (independent) polynomials are 2, 4, \ldots, 2n and the exponents of the corresponding Lie group are 1, 2, \ldots, 2n - 1.

We look for a bracket $\pi_3$ which satisfies

$$\pi_3 \nabla H_2 = \pi_1 \nabla H_4. \quad (4.6)$$

Using trial and error, we end up with the following homogeneous cubic bracket $\pi_3$.

$$\begin{align*}
\{a_i, a_{i+1}\} &= a_i a_{i+1} b_{i+1} , \\
\{a_i, b_i\} &= -a_i b_i^2 - a_i^3, \quad i = 1, \ldots, n - 1, \\
\{a_n, b_n\} &= -a_n b_n^2 - 2a_n^3, \\
\{a_i, b_{i+2}\} &= a_i a_{i+1}^2, \\
\{a_i, b_{i+1}\} &= a_i b_{i+1}^2 + a_i^3, \\
\{a_i, b_{i-1}\} &= -a_{i-2} b_{i-1} a_i, \\
\{b_i, b_{i+1}\} &= 2a_i b_i (b_i + b_{i+1}).
\end{align*} \quad (4.7)$$

We summarize the properties of this new bracket in the following:
**Theorem 4.** The bracket \( \pi_3 \) satisfies:

1) \( \pi_3 \) is Poisson.
2) \( \pi_3 \) is compatible with \( \pi_1 \).
3) \( H_{2i} \) are in involution.

Define \( N = \pi_3 \pi_1^{-1} \). Then \( N \) is a recursion operator. We obtain a hierarchy

\[
\pi_1, \pi_3, \pi_5, \ldots
\]

consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.

4) \( \pi_{j+2} \) grad \( H_{2i} = \pi_j \) grad \( H_{2i+2} \), \( \forall \ i, j \).

The proof of 1) is a straightforward verification of the Jacobi identity. Since \( \pi_3 \) is the Lie derivative of \( \pi_1 \) in the direction of a master symmetry, we also have 2). Since (4.6) holds, 4) follows from properties of the recursion operator. 3) is a consequence of 4).

**A recursion operator for \( C_n \) Toda systems**

We now consider \( C_n \) Toda systems. We will prove that these systems also possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets as in the \( B_n \) case.

The Hamiltonian for \( C_n \) is

\[
H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + e^{q_i - q_{i+1}} + \ldots + e^{q_{n-1} - q_n} + e^{2q_n}.
\]  \( (4.8) \)

We make a Flaschka-type transformation

\[
a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \quad a_n = \frac{1}{\sqrt{2}} e^{\frac{1}{2}q_n}, \quad b_i = -\frac{1}{2} p_i.
\]  \( (4.9) \)

These equations can be written as a Lax pair \( \hat{L} = [B, L] \), where \( L \) is the matrix

\[
\begin{pmatrix}
    b_1 & a_1 & & \\
    a_1 & & & \\
    & \ddots & & \\
    & & & a_{n-1} \\
    & & a_n & b_n \\
    & b_n & a_n & -b_n \\
    & & -b_n & a_n \\
    & & & \ddots \\
    & & & & -a_{n-1} \\
    & & & & & -a_1 \\
    & & & & & -b_1
\end{pmatrix}
\]  \( (4.10) \)

and \( B \) is the skew-symmetric part of \( L \).

In the new variables \( a_i, b_i \) the symplectic bracket \( \pi_1 \) is given by

\[
\{a_i, b_i\} = -a_i, \quad i = 1, 2, \ldots, n - 1,
\]

\[
\{a_i, b_{i+1}\} = a_i, \quad i = 1, 2, \ldots, n - 1,
\]

\[
\{a_n, b_n\} = -2a_n.
\]  \( (4.11) \)

The invariant polynomials for \( C_n \), which we denote by

\[ H_2, H_4, \ldots, H_{2n} \]

are defined by \( H_{2i} = \frac{1}{2i} \text{Tr} L^{2i} \).
We look for a bracket $\pi_3$ which satisfies
\[ \pi_3 \nabla H_2 = \pi_1 \nabla H_4. \] (4.12)

We obtain the following homogeneous cubic bracket $\pi_3$:
\[
\begin{align*}
\{a_i, a_{i+1}\} &= a_i a_{i+1} b_{i+1}, \quad i = 1, 2, \ldots, n - 2, \\
\{a_{n-1}, a_n\} &= 2 a_{n-1} a_n b_n, \\
\{a_i, b_i\} &= -a_i b_i^2 - a_i^3, \quad i = 1, 2, \ldots, n - 1, \\
\{a_n, b_n\} &= -2 a_n b_n^2 - 2 a_n^3, \\
\{a_i, b_{i+2}\} &= a_i a_{i+1}^2, \\
\{a_i, b_{i+1}\} &= a_i b_i^2 + a_i^3, \\
\{a_{n-1}, b_n\} &= a_{n-1}^3 + a_{n-1} b_n^2 - a_{n-1} a_n^2, \\
\{a_i, b_{i-1}\} &= -a_i^2 - a_i b_i, \\
\{a_n, b_{n-1}\} &= -2 a_n^2 a_n, \\
\{b_i, b_{i+1}\} &= 2a_i^2 (b_i + b_{i+1}).
\end{align*}
\] (4.13)

We summarize the properties of this new bracket in the following:

**Theorem 5.** The bracket $\pi_3$ satisfies:

1) $\pi_3$ is Poisson.
2) $\pi_3$ is compatible with $\pi_1$.
3) $H_{2i}$ are in involution.

Define $N = \pi_3 \pi_1^{-1}$. Then $N$ is a recursion operator. We obtain a hierarchy
\[ \pi_1, \pi_3, \pi_5, \ldots \]
consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.

4) $\pi_{j+2} \text{ grad } H_{2i} = \pi_j \text{ grad } H_{2i+2}, \quad \forall i, j$.

The proofs are precisely the same as in the case of $B_n$.

**Master symmetries**

We would like to end with some observations concerning master symmetries. We examine in detail the case of $C_n$. Of course the $B_n$ case is similar.

Due to the presence of a recursion operator, we will use the approach of Oewel. We define $Z_0$ to be the Euler vector field
\[ Z_0 = \sum_{i=1}^{n} a_i \frac{\partial}{\partial a_i} + b_i \frac{\partial}{\partial b_i}. \]

We define the master symmetries $Z_i$ by:
\[ Z_i = D^i Z_0. \]

For obvious reasons I will use the notation $X_0 = Z_0$, $X_2 = Z_1$ and in general $X_{2i} = Z_i$. I will also define $J_i = \pi_{2i+1}$. One calculates easily that
\[ L_{X_0} \pi_1 = \pi_1, \quad L_{X_0} \pi_3 = -\pi_3, \quad L_{X_0} H_2 = 2H_2. \]
Therefore, \( X_0 \) is a conformal symmetry for \( \pi_1, \pi_3, H_2 \). The constants appearing in Oevel's theorem are \( \lambda = 1, \mu = -1 \) and \( \nu = 2 \). Therefore, we obtain

\[
L_{X_2} \pi_1 = L_{Z_1} J_0 = 3 J_1 = 3 \pi_3.
\]

We also have \( X_2(H_2) = 4 H_1 \) and more generally:

\[
X_2(H_j) = (j + 2) H_{j+2}.
\]

The master symmetry \( X_2 \) for the case \( n = 3 \) (system of type \( C_3 \)) is given by

\[
X_2 = \sum_{i=1}^{n} A_i \frac{\partial}{\partial a_i} + B_i \frac{\partial}{\partial b_i},
\]

where

\[
A_1 = \frac{1}{2} a_1(2a_1^2 + 5b_1^2 - b_2^2 - a_2^2 + 2b_1 b_2),
\]
\[
A_2 = \frac{1}{2} a_2(5a_1^2 + 2a_2^2 + 3b_2^2 + b_3^2 + a_3^2 - 2b_1 b_2 + 2b_1 b_3 + 2b_2 b_3),
\]
\[
A_3 = a_3(3a_2^2 + b_2^2 + a_3^2 - 2b_1 b_3 - 2b_2 b_3),
\]
\[
B_1 = b_1^3 - a_1^2 b_1 - 2a_1^2 b_2,
\]
\[
B_2 = 5a_1^2 b_2 + 4a_2^2 b_1 + a_2^2 b_2 + 2a_3^2 b_1 + b_3^2,
\]
\[
B_3 = 2a_1^2 b_2 + 3a_2^2 b_3 - 2a_2^2 b_1 + 2a_3^2 b_1 + 2a_3^2 b_2 + b_3^3 + a_3^2 b_3.
\]

It is interesting to note that one can obtain this master symmetry \( X_2 \) by using the matrix equation

\[
\dot{L} = [B, L] + L^3,
\]

where \( L \) is the Lax matrix (4.10) and \( B \) is the skew-symmetric matrix defined as follows:

\[
b_{12} = -\frac{3}{2} a_1(b_1 + b_2) = -b_{56}, \quad b_{23} = \frac{1}{2} a_2(2b_1 - b_2 - b_3) = -b_{45},
\]
\[
b_{13} = -\frac{3}{2} a_1 a_2 = -b_{46}, \quad b_{24} = -\frac{1}{2} a_2 a_3 = -b_{35},
\]
\[
b_{34} = a_3(b_1 + b_2),
\]

and \( b_{14} = b_{15} = b_{16} = b_{25} = b_{26} = b_{36} = 0 \).

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**References**

